AD NUMBER
AD230349
LIMITATION CHANGES
TO: **Approved for public release; distribution is unlimited.**
FROM: **Distribution authorized to U.S. Gov't. agencies and their contractors; Administrative/Operational Use; JUL 1959. Other requests shall be referred to Ballistic Research Labs., Aberdeen Proving Ground, MD.**
AUTHORITY
USARL ltr, 24 Jan 2003

BRL

REPORT NO. 1065

JULY 1959

A GENERAL ANALYTICAL SOLUTION TO THE PROBLEM OF PHOTOGRAMMETRY

HELLMUT H. SCHMID

DEPARTMENT OF THE ARMY PROJECT NO. 503-06-0110
ORDNANCE RESEARCH AND DEVELOPMENT PROJECT NO. TB3-0538

BALLISTIC RESEARCH LABORATORIES

ABERDEEN PROVING GROUND, MARYLAND

Destroy when no longer needed. DO NOT RETURN

B A L L I S T I C   R E S E A R C H   L A B O R A T O R I E S

REPORT NO. 1065

JULY 1959

A GENERAL ANALYTICAL SOLUTION TO THE PROBLEM OF PHOTOGRAMMETRY

Hellmut H. Schmid

Department of the Army Project No. 503-06-0110
Ordnance Research and Development Project No. TB3-0538

A B E R D E E N   P R O V I N G   G R O U N D ,   M A R Y L A N D

TABLE OF CONTENTS

B A L L I S T I C   R E S E A R C H   L A B O R A T O R I E S

REPORT NO. 1065

HHSchmid/dp
Aberdeen Proving Ground, Md.
July 1959

A GENERAL ANALYTICAL SOLUTION TO THE PROBLEM OF PHOTOGRAMMETRY

ABSTRACT

An analytical treatment of the general problem of photogrammetry is developed which only requires that object point, center of projection and corresponding image point are collinear for any one ray present in a specific photogrammetric measuring procedure. The corresponding formulas, expressing the geometrical relation existing between the spatial coordinates of the object, the plate coordinates of the corresponding image and the elements of orientation, are the bases for a rigorous least squares adjustment, whereby both the image-coordinates measured on a comparator, and the given control data may be considered as erroneous. The corresponding system of normal equations, which is shown to be typical for any photogrammetric measuring problem, is used to form a system of reduced normal equations, the unknowns of which are either the elements of orientation or the spatial coordinates of the model. The application of matrix calculus not only simplifies the presentation but reduces the "bookkeeping effort", while establishing the corresponding program for electronic computers. A special chapter deals with the problem of incorporating additional geometric conditions as they may exist between any one or all of the unknowns of the solution. The application of the described method for the special cases of "strip and block triangulation" is treated, whereby attention is given to the problem of solving the corresponding system of reduced normal equations. For an unlimited strip, a rigorous solution is presented, which is based on the stepwise elimination of certain groups of the unknowns. Finally, an iterative solution is given which makes use of the relaxation method.

I. INTRODUCTION

Theoretical and practical work concerned with the development and testing of analytical solutions for the reduction of photogrammetric records has, in the past, proven the feasibility of such a method. Such a project was initiated at the Ballistic Research Laboratories, Aberdeen Proving Ground, Maryland with the study of the orientation of a single camera followed by a solution for the problem of triangulation by a combination of two photogrammetric cameras. These two phases have been presented in the BRL Reports No. 880 [1]* and No. 961 [2]. The phase logically to follow is the development of an analytical solution for the n-camera case. The problem of triangulating, for example, the space-time coordinates of points of trajectories recorded simultaneously on more than two photogrammetric cameras belongs in this category, as well as the problems of strip and block triangulation.

In [2] an approach was outlined which would have made it possible to extend the two camera solution to the treatment of the n-camera problem. The most serious objection to that approach is the fact that, in such a case, the matrix, associated with the vectors of the residuals, loses more and more its diagonal character. Thus, the basis for the feasibility of the corresponding numerical solution is being impaired. (Compare [2], page 22 and schematics on page 48). Furthermore, the solution becomes cumbersome if additional information must be introduced expressing certain geometric conditions concerned with the coordinates of the points of the object to be triangulated. Last but not least, the "bookkeeping effort" in the preparation of the electronic computations would have been considerable, due to the necessity of distinguishing between various kinds of control data and corresponding conditional equations, which, to make things worse, are of somewhat different character, depending on the number of cameras involved in any one specific triangulation.

The following solution overcomes these objections. As an additional feature, this solution is based in its entirety on the simplest mathematical presentation conceivable. The use of matrix calculus for setting up a system of reduced normal equations is especially suited for electronic computers.

* Reference at the end of the paper.

The following presentation of the solution is complete with respect to derivation of the necessary formulas, differential coefficients and auxiliaries. However, with respect to general remarks and basic principles, referring to photogrammetric as well as to statistical subjects, reference will occasionally be made to [1] and [2] which are considered to be available to the casual reader.

II. THE GENERAL PROBLEM OF ANALYTICAL PHOTOGRAMMETRY

The general problem of analytical photogrammetry may be defined as the simultaneous restitution of the orientations of any number of photogrammetric records and the reconstruction of the object space by triangulating corresponding rays. There must be no limitations as to the type and orientation of any one camera, so long as the corresponding bundle of rays corresponds to the principle of a central perspective. No limitations must be made with respect to the number, type and location of control data including absolute given, partial given and relative points, so long as the given information satisfies at least the geometrical requirements for a unique solution.

In a general solution it must be possible to enforce any number of geometric conditions concerned with any one or all orientation parameters, as well as with any one or all coordinates of the points of the object space. Furthermore, it must be possible to consider both the plate measurements and the given control data as erroneous, whereby the computation must allow the introduction of individual weighting factors. Finally, a general solution must derive, from the treatment of the redundant information, such expressions of precision which will give information about the mean error of an observation of unit weight - the mean errors of the orientation elements as well as of the triangulated coordinates.

III. THE CONDITION OF CO-LINEARITY AS THE SOLE CONDITION NECESSARY TO SOLVE THE PROBLEM OF ANALYTICAL PHOTOGRAMMETRY

For later reference, a description of a few fundamental photogrammetric operators seems appropriate.

A. The photogrammetric bundle and its orientation

By connecting any identifiable detail on the photographic plate with a point situated outside of the plane of the photograph, it is possible to construct a bundle of rays. (Fig. 1) The point, thus common to all rays, is the so called center of projection, denoted by O. It is hardly possible, and for our purposes certainly unnecessary, to interpret this point physically. The position of this point relative to the photographic plate is fixed if, for example, an arbitrarily oriented rectangular plate coordinate system (x and y) is introduced in which the base of the perpendicular to the plane of the plate through O, denoted by P (principal point), has the coordinates x_p and y_p. The length of this perpendicular, denoted by c, is a scale factor and must not be physically interpreted. The sole purpose of this phase of the photogrammetric evaluation technique is to provide means suited for an unambiguous construction of a bundle of rays. This is accomplished by measuring x, y plate coordinates of identifiable detail on the photograph in the same plate coordinate system, in which the point P is being described by its parameters x_p and y_p. The parameters c, x_p and y_p are commonly referred to as the elements of interior orientation.

After such a bundle is obtained, the problem is to orient it unambiguously in space, (1) by assigning to the point O, a specific spatial position expressed, for example, by three linear parameters, X_o, Y_o and Z_o with respect to an arbitrarily established Cartesian spatial coordinate system; (2) by defining with respect to the axes of this coordinate system, the direction of the vector as formed by the extension of the line $\overline{PO}$ into space by two rotational components, (e.g., α and ω rotations), and (3) by establishing the spatial position of the plate coordinate system by a third rotation (swing angle κ) around the vector described, and positioned according to (2).

The parameters X_o, Y_o, Z_o, α, ω, κ are commonly referred to as the elements of exterior orientation.

Both groups, the elements of interior and the elements of exterior orientation, shall from now on be considered jointly if reference is made simply to the elements of orientation. The notation **O** is used if all

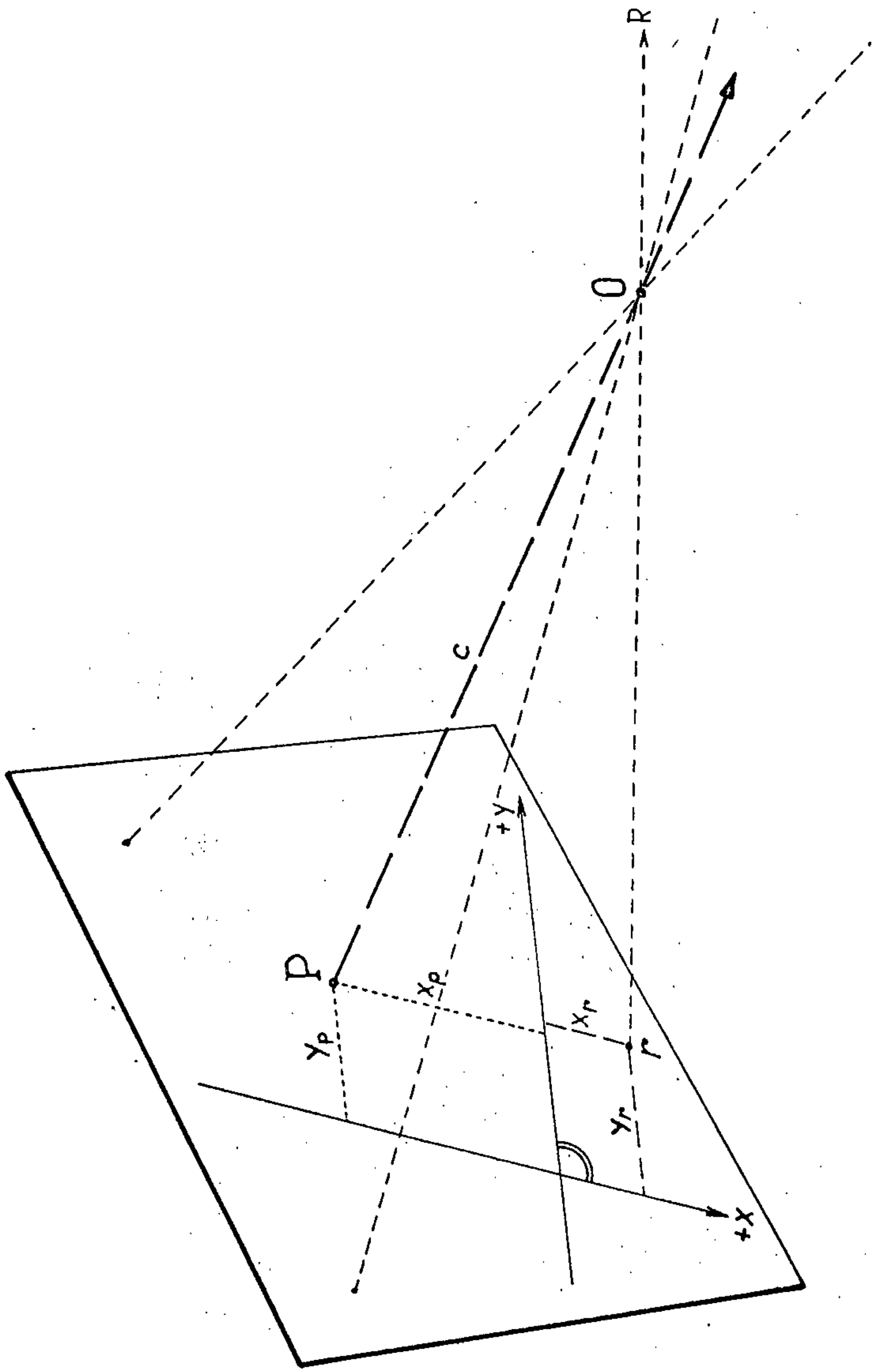


Fig. I

orientation elements are being considered present in a specific photogrammetric measuring system, while the notation O_i is used if only a group associated with a certain camera station I is being regarded.

B. The control data and the associated residuals.

It is common practice to refer to points known with their absolute position as absolute control points, to points known with either one or two positioning parameters as partial control points, and to points not known at all with respect to their absolute position, as relative control points. Because all these points belong to the object photographed, it is conceivable to refer to them in their totality simply as control points, and denote their spatial positions by $\mathbf{X}$, while the spatial location of any one individual point J shall be denoted by $\mathbf{X}_j$. In case the given control point coordinates are not flawless, it will become necessary in a general solution to treat these parameters like measurements and allow for corresponding corrections in the least squares solution. The designation $\mathbf{V}$ refers to the corrections of all control data and correspondingly $\mathbf{V}_j$ is being used if only a specific control point "J" is considered.

C. The plate measurements and the associated residuals.

As described in (A), any individual ray within a photogrammetric bundle is unambiguously positioned by two plate measurements x and y. In analogue to the above notation, we shall denote all plate coordinates involved in a specific photogrammetric measuring system by $\mathbf{x}$ and the plate coordinates of a specific image point, on a specific plate by $\mathbf{x}_{ij}$. Correspondingly, we denote the total number of plate measurements and their residuals by $\boldsymbol{\ell}$ and $\mathbf{v}$ respectively, and the plate measurements and their residuals associated with a specific point, on a specific photograph by $\boldsymbol{\ell}_{ij}$ and $\mathbf{v}_{ij}$.

The object space (the model) can be visualized as the integrated effect produced by the intersections of corresponding rays, each of which is positioned in a specific oriented photogrammetric bundle according to the corresponding image location as stored on the specific photographic record.

With the notation introduced above we may consequently write:

$$X = \phi(O, x) \tag{1}$$

Formula (1) expresses the general problem of photogrammetry, indicating a functional relation between the model X , the plate coordinates x and the orientation parameters O . Introducing the plate measurements, their residuals and the residuals of the given control data, we obtain, with respect to (1), a system of observational equations of the form:

$$f(v, V) = F(O, X, \ell) \tag{2}$$

The roots of the corresponding normal equation system represent the numerical solution of the general photogrammetric problem. The result, as expressed by formula (1), has been explained as an integrated effect of the contribution of all the individual rays. Because no one ray distinguishes itself basically from any other ray, this interpretation suggests that for an individual ray, a corresponding functional relation exists which is obtained by simply indexing the corresponding parameters, thus, leading, according to formula (2), to the corresponding observational equations:

$$f(v_{ij}, V_j) = F(O_i, X_j, \ell_{ij}) \tag{3}$$

To comprehend this result we recollect that the bundle of rays of an idealized physical photogrammetric camera has its geometrical representation in the concept of the central perspective. Any photogrammetric bundle can thus be considered as a population, the members of which are the individual rays. Any algebraic expression representing a single ray of such a bundle, may thus be envisaged as representing, collectively, the bundle in its entirety, by simply omitting the index denoting the specific ray.

Because any one photogrammetric bundle is based on the concept of the central perspective, and any one photogrammetric problem may be considered as a combination of any number of such bundles, it follows that an equation representing the geometrical properties of an individual ray can be considered as adequate to express, collectively, the problem of analytical photogrammetry.

Consequently, it must be possible to develop an analytical solution for the most general problem of photogrammetry, based solely on formulas which express the geometrical properties of an individual ray belonging to a bundle of rays, formed according to the concept of the central perspective.

The corresponding relation is the condition that the center of projection 0, the image point r and the object point R are collinear. (Fig. 2)

From Figure 2 we obtain:

$$\vec{R} = \mu \cdot \vec{r}, \quad \text{where } \mu \text{ is a scale factor} \tag{4}$$

The projection of the vectors $\vec{r}$ and $\vec{R}$ respectively into the three coordinate planes gives the component equations:

$$\begin{aligned} X &= X_o + \mu u \\ Y &= Y_o + \mu v \\ Z &= Z_o + \mu w \end{aligned} \tag{5}$$

The triplet of formulas (5) is the analytical expression for the condition that the points 0, r and R lie on a straight line.

By eliminating the scale factor μ in formulas (5) we obtain:

$$\text{where } (X) = X - X_o$$

$$(X) = (Z)\frac{u}{w}$$

$$(Y) = Y - Y_o \tag{6}$$

$$(Y) = (Z)\frac{v}{w}$$

$$(Z) = Z - Z_o$$

From Fig. 2 we read directly:

$$\vec{r} = iu + jv + kw = \hat{i}\bar{x} + \hat{j}\bar{y} + \hat{k}c \tag{7}$$

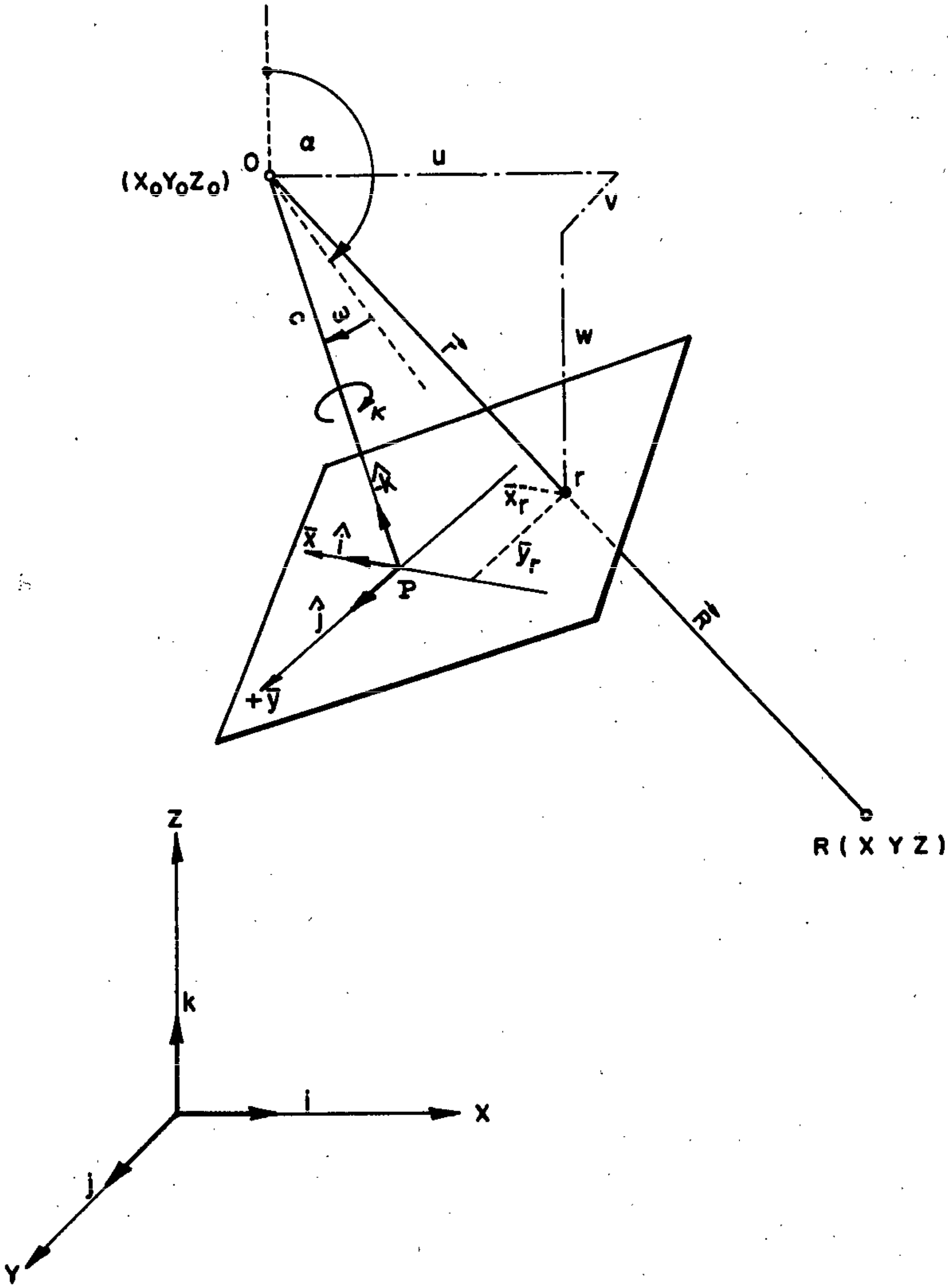

α
O
u
$(X_0 Y_0 Z_0)$
v
w
c
ω
$\vec{r}$
κ
r
$\bar{x}_r$
$\bar{x}$
$\hat{i}$
$\bar{y}_r$
P
$\hat{j}$
$\vec{R}$
$+\bar{y}$
R (X Y Z)
Z
k
i
X
j
Y

FIGURE 2

whereby the transformation matrix of the two vector triplets is:

$$\begin{array}{c|c} & \begin{array}{ccc} \hat{i} & \hat{j} & \hat{k} \end{array} \\ \hline \begin{array}{c} i \\ j \\ k \end{array} & \begin{pmatrix} i\hat{i} & i\hat{j} & i\hat{k} \\ j\hat{i} & j\hat{j} & j\hat{k} \\ k\hat{i} & k\hat{j} & k\hat{k} \end{pmatrix} \end{array} \tag{8}$$

Denoting, for example, the two rotational angles between the two vector triplets according to Figure 2 with α and ω, we obtain:

$$\begin{aligned} u &= \bar{x} \cos\alpha - \bar{y} \sin\alpha \sin\omega + c \sin\alpha \cos\omega \\ v &= \bar{y} \cos\omega + c \sin\omega \\ w &= -\bar{x} \sin\alpha - \bar{y} \cos\alpha \sin\omega + c \cos\alpha \cos\omega \end{aligned} \tag{9}$$

From Figure 3 we obtain:

$$\begin{aligned} \bar{x} &= -(x - x_p) \cos\kappa - (y - y_p) \sin\kappa \\ \bar{y} &= -(x - x_p) \sin\kappa + (y - y_p) \cos\kappa \end{aligned} \tag{10}$$

where x and y are the plate coordinates of an image point in an arbitrarily oriented rectangular reference system (plate coordinate system).

Substituting (10) into (9) and using (6) we have:

$$\begin{aligned} X &= \frac{(Z)\left[(x-x_p)A_1 + (y-y_p)A_2 + c\,D\right]}{Q} + X_o \\ Y &= \frac{(Z)\left[(x-x_p)B_1 + (y-y_p)B_2 + c\,E\right]}{Q} + Y_o \end{aligned} \tag{11}$$

with $Q = (x-x_p)C_1 + (y-y_p)C_2 + c\,F$

and

$$\begin{aligned} x &= \frac{c\left[(X)A_1 + (Y)B_1 + (Z)C_1\right]}{q} + x_p \\ y &= \frac{c\left[(X)A_2 + (Y)B_2 + (Z)C_2\right]}{q} + y_p \end{aligned} \tag{12}$$

with $q = (X)\,D + (Y)\,E + (Z)\,F$

xy-plane as diapositive seen from O

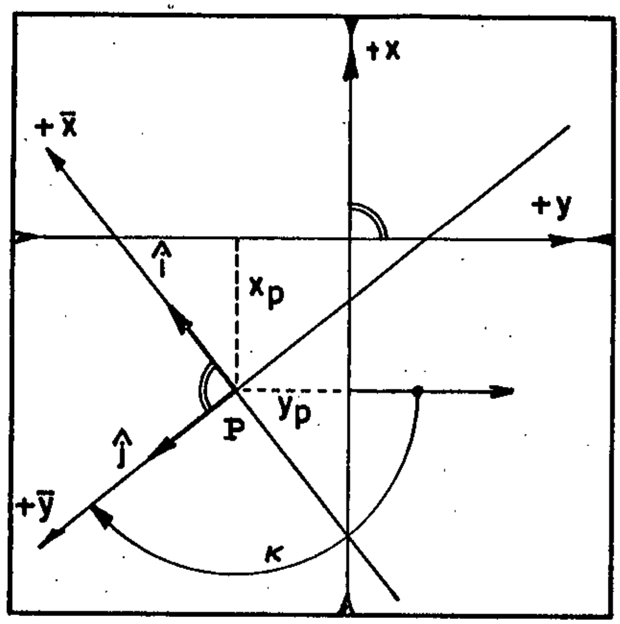


FIGURE 3

where we have denoted as essentially direction cosines:

$$
\begin{aligned}
A_1 &= -\cos\alpha\cos\kappa + \sin\alpha\sin\omega\sin\kappa \\
B_1 &= -\cos\omega\sin\kappa \\
C_1 &= \sin\alpha\cos\kappa + \cos\alpha\sin\omega\sin\kappa \\
A_2 &= -\cos\alpha\sin\kappa - \sin\alpha\sin\omega\cos\kappa \\
B_2 &= \cos\omega\cos\kappa \\
C_2 &= \sin\alpha\sin\kappa - \cos\alpha\sin\omega\cos\kappa \\
D &= \sin\alpha\cos\omega \\
E &= \sin\omega \\
F &= \cos\alpha\cos\omega
\end{aligned}
\tag{13}
$$

Each of the pair of formulas (11) and (12) represent the algebraic expression for the geometrical condition of co-linearity of the points O, r and R, solved explicitly for the coordinates of point R in the i, j, k system and of point r in the $\hat{i}$, $\hat{j}$, $\hat{k}$ system, respectively. The symmetrical arrangement of the formulas is a direct consequence of the reversibility of any central perspective. (compare [1] page 8)

In [2] the algebraic solution was continued by eliminating such X_i, Y_i and Z_i's from the solution which were not given as control data. This step led, for relative control points, to an algebraic expression for the condition of intersection, and correspondingly for partially given control points, to the condition of intersection at either one or two given control coordinates. As a direct consequence of that approach, different types of conditional equations were obtained for the different types of control points. To complicate the situation further, the number of any particular set of such conditional equations depends on the number of camera stations involved in any specific triangulation problem.

The most serious objection against the approach chosen in [2] results from the fact that, for the condition of intersection, it is necessary to have at least four, and under certain conditions even six plate coordinate measurements together with their residuals in any single conditional equation. Furthermore, in the case of multi-camera triangulation, the same residuals appear in more than one conditional equation, making a rigorous least squares treatment laborious, or in more complex cases, even impractical.

The coordinates of the relative and partial control points not given at the outset of the computations become known in the solution, as presented in [2] , only implicitily as functions of the adjusted plate measurements and the orientation elements. The introduction of any additional geometric conditions as they may exist for any one or all of these coordinates requires complex mathematical manipulations, prohibitive from the standpoint of computing economy.

These difficulties have been avoided in the present solution by including the elimination of the unknown coordinates of the model in the process of the numerical treatment. In other words, instead of eliminating algebraically the unknown X_i, Y_i and Z_i's, beforehand, in the system of formulas (12), these quantities are carried as unknowns together with the orientation elements and are solved for during the numerical solution.

All of the beneficial consequences of this approach will become evident in the least squares solution as described in the next chapter. It suffices to mention here the most important features:

(1) Each point, without regard to its character as absolute, partial or relative control point is being treated alike, thus giving rise to two equations, of the form of formulas (12), for each camera station at which the point was recorded. Any given information for example, in the form of spatial coordinates, is introduced by simply eliminating the corresponding parameter corrections from the sequence of unknowns in the least squares solution. Therefore, all points are treated with only one basic set of equations, (formulas (12)), which are explicit in terms of the measured plate coordinates x and y, respectively.

(2) The fact that the coordinates of the points of the model, to be triangulated are carried as unknowns in the solution, makes it possible to introduce readily any additional existing geometric conditions for any one or all of these coordinates.

IV. THE LEAST SQUARES SOLUTION

The formulas (12) express the plate coordinates x and y as functions of the orientation elements and the spatial coordinates of the corresponding object point. For any one point J photographed at a certain camera station I, we may therefore write in general terms, according to formula (3):

$$\begin{aligned} \ell_{x_{ij}} + v_{x_{ij}} &= F_1 \left[(\alpha, \omega, \kappa, X_o, Y_o, Z_o, c, x_p)_i, \ (X + V_X, \ Y + V_Y, \ Z + V_Z)_j \right] \\ \ell_{y_{ij}} + v_{y_{ij}} &= F_2 \left[(\alpha, \omega, \kappa, X_o, Y_o, Z_o, c, y_p)_i, \ (X + V_X, \ Y + V_Y, \ Z + V_Z)_j \right] \end{aligned} \tag{14}$$

From the Taylor expansion for the right hand side of the equations (14), neglecting terms of second and higher order, we obtain the observational equations:

$$\begin{aligned} v_{x_{ij}} - \left(\frac{\partial F_1}{\partial X} \frac{q}{c} \dot{V}_X + \frac{\partial F_1}{\partial Y} \frac{q}{c} \dot{V}_Y + \frac{\partial F_1}{\partial Z} \frac{q}{c} \dot{V}_Z \right)_j &= \left(\frac{\partial F_1}{\partial \alpha} \Delta \alpha + \frac{\partial F_1}{\partial \omega} \Delta \omega + \frac{\partial F_1}{\partial \kappa} \Delta \kappa + \frac{\partial F_1}{\partial X_o} \Delta X_o \right. \\ &\quad \left. + \frac{\partial F_1}{\partial Y_o} \Delta Y_o + \frac{\partial F_1}{\partial Z_o} \Delta Z_o + \frac{\partial F_1}{\partial c} \Delta c + \frac{\partial F_1}{\partial x_p} \Delta x_p \right)_i + \left(\frac{\partial F_1}{\partial X} \Delta X + \frac{\partial F_1}{\partial Y} \Delta Y + \frac{\partial F_1}{\partial Z} \Delta Z \right)_j - \Delta \ell_{x_{ij}} \\ v_{y_{ij}} - \left(\frac{\partial F_2}{\partial X} \frac{q}{c} \dot{V}_X + \frac{\partial F_2}{\partial Y} \frac{q}{c} \dot{V}_Y + \frac{\partial F_2}{\partial Z} \frac{q}{c} \dot{V}_Z \right)_j &= \left(\frac{\partial F_2}{\partial \alpha} \Delta \alpha + \frac{\partial F_2}{\partial \omega} \Delta \omega + \frac{\partial F_2}{\partial \kappa} \Delta \kappa + \frac{\partial F_2}{\partial X_o} \Delta X_o \right. \\ &\quad \left. \frac{\partial F_2}{\partial Y_o} \Delta Y_o + \frac{\partial F_2}{\partial Z_o} \Delta Z_o + \frac{\partial F_2}{\partial c} \Delta c + \frac{\partial F_2}{\partial y_p} \Delta y_p \right)_i + \left(\frac{\partial F_2}{\partial X} \Delta X + \frac{\partial F_2}{\partial Y} \Delta Y + \frac{\partial F_2}{\partial Z} \Delta Z \right)_j - \Delta \ell_{y_{ij}} \end{aligned} \tag{15}$$

where

$$- \Delta \ell_{x_{ij}} = x^o_{ij} - \ell_{x_{ij}}$$

$$- \Delta \ell_{y_{ij}} = y^o_{ij} - \ell_{y_{ij}}$$

The $\dot{V}$ are the normalized V-residuals according to the formula:

$$V = \overset{\circ}{V} \frac{q}{c} \tag{16}$$

x_{ij}^{o} and y_{ij}^{o} are computed with formulas (12) and the approximation values of the unknowns which are denoted by "o". It is obvious that in any one specific set of observational equations may appear either certain coordinate corrections Δ_X or the corresponding residual errors $\dot{V}$ depending on the character of the object point under consideration. With the introduction of the $\dot{V}$-residuals, according to formula (16), each ray has been assigned a specific group of such residuals. Such an approach is desirable from the numerical standpoint, because any correlation is avoided between the coefficients of the matrix associated with the residuals of the various rays, intersecting at a specific point. From the theoretical standpoint such a solution seems to be advantageous because the individual bundles of rays will conform to the pattern of the control data without undue restraint. In paragraph (D) of this chapter the possibility will be discussed of arranging the least squares adjustment in such a way that for each given control coordinate only one specific residual V is obtained, independent of the number of rays intersecting at the point under consideration.

The relation between the approximation values and the final values of the unknowns is given by:

$$\begin{aligned}
\alpha &= \alpha^{o} + \Delta\alpha & c &= c^{o} + \Delta c \\
\omega &= \omega^{o} + \Delta\omega & x_p &= x_p^{o} + \Delta x_p \\
\kappa &= \kappa^{o} + \Delta\kappa & y_p &= y_p^{o} + \Delta y_p \\
X_o &= X_o^{o} + \Delta X_o & X_j &= X_j^{o} + \Delta X_j \\
Y_o &= Y_o^{o} + \Delta Y_o & Y_j &= Y_j^{o} + \Delta Y_j \\
Z_o &= Z_o^{o} + \Delta Z_o & Z_j &= Z_j^{o} + \Delta Z_j
\end{aligned} \tag{17}$$

A. A Direct Solution

Using matrix notation, the system of observational equations for an m-ray solution, according to formulas (3) or (15), may be written as:

$$\begin{bmatrix} A_1 & & & & & \\ & A_2 & & & & \\ & & A_3 & & & \\ & & & \bullet & & \\ & & & & \bullet & \\ & & & & & A_m \end{bmatrix} \begin{bmatrix} V_1 \\ V_2 \\ V_3 \\ \bullet \\ \bullet \\ V_m \end{bmatrix} = \begin{bmatrix} B_1 \\ B_2 \\ B_3 \\ \bullet \\ \bullet \\ B_m \end{bmatrix} \begin{bmatrix} \Delta_1 \\ \Delta_2 \\ \Delta_3 \\ \bullet \\ \bullet \\ \Delta_m \end{bmatrix} - \begin{bmatrix} \ell_1 \\ \ell_2 \\ \ell_3 \\ \bullet \\ \bullet \\ \ell_m \end{bmatrix} \qquad (18)$$

The A_i are the coefficient matrices of the corresponding residual vectors V_i. In case the absolutely given control data are considered flawless

$$A_1 = A_2 = A_3 = \bullet \bullet \bullet = A_m = I$$

the unit matrix. The B_i are the coefficient matrices of the vectors of the corresponding parameter corrections Δ_i. The ℓ_i are the vectors of the absolute terms of the observational equations. We may rewrite the system of observational equations with obvious notation as $Av = B\Delta - \ell$ (19)
with the weights P

Assuming the observations to be independent and normally distributed the most probable values of the unknowns are obtained by minimizing $v^T P v$, where P denotes the weight matrix

$$P = \begin{bmatrix} P_1 & & & & & \\ & P_2 & & & & \\ & & P_3 & & & \\ & & & \bullet & & \\ & & & & \bullet & \\ & & & & & P_m \end{bmatrix} \qquad (20)$$

An individual p_i matrix designates the weights of the plate measurements and of the control data, as they pertain to a specific ray. The introduction of weighting factors for the plate measurements may become necessary to express varying degrees of precision associated with the original observations. Such variations may arise from the method of measuring, the varying image quality caused by loss of definition towards the edges of the photograph, or from a decrease of accuracy with which the distortion correction is known for image points at increasing radial distances from the center of the plate. Various degrees of dependability of the given control data can be considered, as well, by the introduction of corresponding weighting factors. Infinitely large weight with respect to the given control data will eliminate the V from the solution, thus distributing the unavoidable discrepancies among the plate coordinate measurements only; however, infinitely large weight assigned to the plate coordinate measurements will make the given control data absorb all the discrepancies present in connection with a certain ray under consideration. [3] While the boundary conditions that were just mentioned are only of secondary importance, the choice of suitable weighting coefficients makes it possible to prevent an undue deformation of the model obtained photogrammetrically by possible strain in the configuration of the given control data. In case the original observations (image point or control coordinates) are not independent from each other, the corresponding correlations can be allowed for by introducing the corresponding correlation coefficients into the P matrix. In case all the p-values are equal, it is convenient to consider P as the unit matrix.

In setting up the corresponding normal equations, one has to take into account the fact that in the most general case each observational equation contains more than one measurement. Furthermore, in such a case, certain measurements and their residuals appear in more than one observational equation. Helmert in [4], (pp 215-222), has shown a direct solution of the general problem of a least squares adjustment. (Compare [2], paragraph IV). Accordingly, we obtain in our case a set of normal equations as shown in formulas (21):

$$\begin{aligned} A P^{-1} A^T K - E\Delta &= -\Omega \\ -B^T K \quad &= 0 \end{aligned} \tag{21}$$

where k denotes a vector of unknown Lagrange multipliers or correlates. A direct solution would be obtained by the inversion of the system (21). That is, however, hardly practical because in addition to the unknown parameter corrections, Δ's, we have for an m-ray problem 2m additional unknown k-values, a fact which increases the number of unknown parameters beyond the means of practical computations.

Next, a method which is commonly used in partitioning, a system of linear equations shall be presented which will be useful for our problem. Assuming a system of linear equations:

$$A x = \ell \tag{22}$$

or consequently

$$x = A^{-1} \ell \tag{23}$$

Such a system can be partitioned in any arbitrary manner leading to the following sub-matrices and sub-vectors:

$$\begin{array}{c|c} A_1 x_1 & + A_{12} x_2 = \ell_1 \\ \hline A_{21} x_1 & + A_2 \; x_2 = \ell_2 \end{array} \tag{24}$$

According to (23) we may write with the notation of (24)

$$x_1 = A_1^{-1} \lambda_1 \qquad \lambda_1 = \ell_1 - A_{12} x_2 \tag{25}$$

and therefore

$$x_1 = A_1^{-1} \; \ell_1 - A_1^{-1} \; A_{12} x_2 \tag{26}$$

Introducing (26) into the lower portion of the system (24) we obtain:

$$\begin{aligned} A_{21} A_1^{-1} \ell_1 - A_{21} A_1^{-1} A_{12} x_2 + A_2 x_2 &= \ell_2 \\ (A_2 - A_{21} A_1^{-1} A_{12}) \; x_2 &= \ell_2 - A_{21} A_1^{-1} \; \ell_1 \end{aligned} \tag{27}$$

which may be written in a form analogous to (23) as:

$$x_2 = A_2^{*-1}\lambda_2 \qquad \text{with} \qquad \begin{aligned} A_2^* &= A_2 - A_{21}A_1^{-1}A_{12} \\ \lambda_2 &= \ell_2 - A_{21}A_1^{-1}\ell_1 \end{aligned} \tag{28}$$

The computation of x_1 is then carried out with formula (26).

Because the sequence of the steps in the process of partitioning is in no way restricted, it is possible to write as in (28) and (26)

$$x_1 = A_1^{*-1}\lambda_1 \qquad \text{with} \qquad \begin{aligned} A_1^* &= A_1 - A_{12}A_2^{-1}A_{21} \\ \lambda_1 &= \ell_1 - A_{12}A_2^{-1}\ell_2 \end{aligned} \tag{29}$$

and

$$x_2 = A_2^{-1}\ell_2 - A_2^{-1}A_{21}x_1 \tag{30}$$

The method just described obviously eliminates one of the two groups of unknowns as chosen by the process of partitioning and solves for the other group. If the method is used to partition a system of normal equations at any point along its diagonal, it follows from the symmetry of such a system that $A_{12}^T = A_{21}$. Further, it can be shown that in such a case the matrix A_2^* in (28) and correspondingly the matrix A_1^* in (29) are again symmetrically arranged square matrices.

Because the matrix $AP^{-1}A^T$ in formula (21) is non-singular in our problem, we may apply the method of partitioning as just described for the purpose of eliminating the k-values from the original normal equation system. The reduced normal equation system is, according to formula (27),

$$\left[B^T(AP^{-1}A^T)^{-1}B\right]\Delta = B^T(AP^{-1}A^T)^{-1}\ell \tag{31}$$

The feasibility of this method of partitioning depends on the effort necessary to invert the matrix. $AP^{-1}A^T$. Because this matrix is even for the most general case of our problem, a sequence of fully separated symmetrically arranged (2 x 2) square sub-matrices, it is possible to accumulate the normal equation system (31) stepwise, as explained in [2] page 22. Thus we obtain:

$$\sum_{i=1}^{m} \left[B^T (AP^{-1} A^T)^{-1} B \right]_i \Delta = \sum_{i=1}^{m} \left[B^T (AP^{-1}A^T)^{-1} \ell \right]_i \qquad (32)$$

whereby m, the number of $AP^{-1}A^T$ submatrices, equals the number of rays present in the specific problem. As already mentioned at the beginning of this paragraph, in case only the residuals of the plate measurements are present the A_i matrices are unit matrices and therefore the $(AP^{-1}A^T)^{-1}$ term in (31) reduces to P. In such a case the system (31) resembles a system of normal equations associated with observational equations for independent indirect measurements. The final normal equation system in such a case can be accumulated stepwise according to formula (32) by considering in each computational step a single observational equation.

After the vector of the Δ corrections of the unknown parameters is computed, we obtain with the first group of equations in formula (21), the k-values.

$$k = (AP^{-1}A^T)^{-1} (B\Delta - \ell) \qquad (33)$$

and the residuals v and $\dot{V}$ by:

$$V = \begin{bmatrix} v_x \\ v_y \\ \dot{V}_X \\ \dot{V}_Y \\ \dot{V}_Z \end{bmatrix} = P^{-1} A^T k \qquad (34)$$

The V-values are then computed with formulas (16).

From a study of formula (31) it becomes evident that the feasibility of the proposed solution depends on the possibility of inverting a more or less large matrix of normal equations. In special applications of photogrammetry, such as laboratory measurements, ballistic measurements, terrestrial applications and, more recently, cadastral surveying, using airborne photography, the single model still plays a dominant role. The total number of unknowns in such cases may very well be within the limits which today can be reduced by inverting the corresponding normal equation matrix. However, depending on the number of relative control points considered essential in order to satisfy the requirement for redundant information and on the type of the electronic computer available, there will arise in many photogrammetric applications the problem of treating numbers of unknown parameters exceeding the computational capacity for reduction by direct inversion. For such cases, it is desirable, however, to maintain the advantages of the systematism and simplicity described earlier in forming the observational equations and the corresponding normal equations. Consequently, the solution to our problem must concern itself with methods for determining the roots of a normal equation system of the type shown in formula (31).

B. A Solution by Partitioning

In order to further reduce the number of unknowns in the normal equation system obtained in formula (31), we split the B matrix and the Δ vector in such a way that one group is associated with the model and the other group with the camera orientations. We denote the corresponding submatrices and subvectors by B_x, B_o and Δ_x and Δ_o, respectively. With these notations we can present the system of normal equations (31) as follows:

$$\begin{array}{l:l}
\left[B_x^T (AP^{-1}A^T)^{-1} B_x\right]\Delta_x & +\left[B_x^T (AP^{-1}A^T)^{-1} B_o\right]\Delta_o = B_x^T (AP^{-1}A^T)^{-1}\ell \\
\hdashline
\left[B_o^T (AP^{-1}A^T)^{-1} B_x\right]\Delta_x & +\left[B_o^T (AP^{-1}A^T)^{-1} B_o\right]\Delta_o = B_o^T (AP^{-1}A^T)^{-1}\ell
\end{array} \tag{35}$$

Normal Equations According to Formula (35)

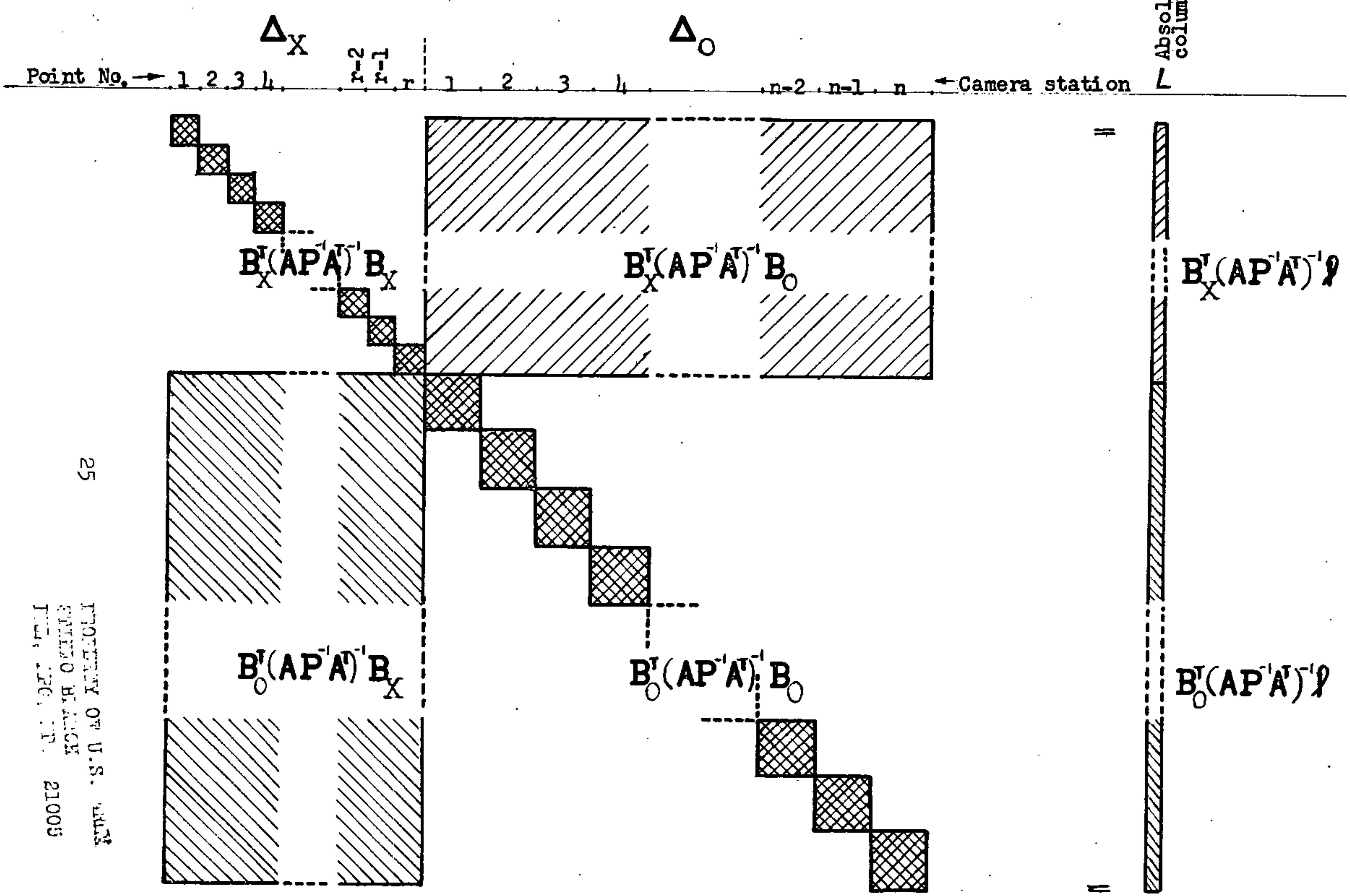


Figure 4

The normal equation system as shown in (35) is typical in its arrangement for any photogrammetric triangulation problem. The number of points recorded at any one station and the number of stations involved in a specific measuring program will obviously influence the overall size of this matrix system but will not change its basic character. A study of the corresponding matrix (see Fig. 4) shows that we have along the diagonal, a sequence of fully separated symmetrically arranged square submatrices. The fact that the two types of submatrices which appear vary in size is less significant than the fact that in each submatrix of the first group, $B_X^T(AP^{-1}A^T)^{-1}B_X$, the spatial coordinates of only one specific object point (in a general case up to 3) are present, while in each submatrix of the second group $B_O^T(AP^{-1}A^T)^{-1}B_O$, only the orientation elements of one specific camera station (in a general case up to 9), appear. The $B_X^T(AP^{-1}A^T)^{-1}B_O$ and the $B_O^T(AP^{-1}A^T)^{-1}B_X$ submatrices express the fact that a specific point was photographed from certain camera stations.

Viewing our system of normal equations with respect to the method of partitioning as described in this report with formulas (22) through (30), a suitable point for partitioning is obviously that point on the diagonal, which separates the parameters associated with the model, from the parameters connected with the camera orientations, as indicated by the dotted lines in formula (35). According to formula (27), we may write:

$$\begin{aligned}&\left\{B_O^T(AP^{-1}A^T)^{-1}B_O-\left[B_O^T(AP^{-1}A^T)^{-1}B_X\right]\left[B_X^T(AP^{-1}A^T)^{-1}B_X\right]^{-1}\left[B_X^T(AP^{-1}A^T)^{-1}B_O\right]\right\}\Delta_O=\\&\left\{B_O^T(AP^{-1}A^T)^{-1}-\left[B_O^T(AP^{-1}A^T)^{-1}B_X\right]\left[B_X^T(AP^{-1}A^T)^{-1}B_X\right]^{-1}\left[B_X^T(AP^{-1}A^T)^{-1}\right]\right\}\ell\end{aligned} \tag{36}$$

or, with reference to (32):

$$\sum_{i=1}^{r}(C_O B_O)_i\,\Delta_O=\sum_{i=1}^{r}(C_O\ell)_i \tag{37}$$

where

$$C_{o_i} = B_o^T \left\{ (AP^{-1}A^T)^{-1} - \left[(AP^{-1}A^T)^{-1} B_x \right] \left[B_x^T (AP^{-1}A^T)^{-1} B_x \right]^{-1} \left[B_x^T (AP^{-1}A^T)^{-1} \right] \right\}_i$$

and r = number of points of the model.

In any photogrammetric reduction problem we may choose to eliminate either the coordinate corrections of the model or the corrections of the orientation parameters in the system of equations (31). Accordingly, we may write as in formula (37)

$$\sum_{i=1}^{n} (C_x B_x)_i \Delta_x = \sum_{i=1}^{n} (C_x \ell)_i \tag{38}$$

where:

$$C_{x_i} = B_x^T \left\{ (AP^{-1}A^T)^{-1} - \left[(AP^{-1}A^T)^{-1} B_o \right] \left[B_o^T (AP^{-1}A^T)^{-1} B_o \right]^{-1} \left[B_o^T (AP^{-1}A^T)^{-1} \right] \right\}_i$$

and n = number of photogrammetric cameras.

In photogrammetric measuring problems, generally speaking, the number of unknown orientation parameters will be less than the number of unknown coordinates of the model; therefore the elimination of Δ_x as suggested with formulas (37) will generally lead to the most economical solution. Block triangulation with a high degree of sidelap may be mentioned as an exception.

With the vector Δ_o known, the vector Δ_x may be computed for each point separately, or vice-versa, from the upper portion of formulas (35). Thus for example Δ_x is:

$$\Delta_x = \left[B_x^T (AP^{-1}A^T)^{-1} B_x \right]^{-1} \left[B_x^T (AP^{-1}A^T)^{-1} \right] (\ell - B_o \Delta_o) \tag{39}$$

with $\Delta = \begin{bmatrix} \Delta x \\ \Delta o \end{bmatrix}$ the $\mathbf{k}$ and the corresponding $\mathbf{v}$ and $\mathbf{V}$ values may be computed with formulas (33), (34) and (16).

However, it appears to be more advisable to compute the coordinates of model points with such computational steps, as outlined in Chapter IV, E. of this report. Such an approach appears to be especially suited because the corresponding computational means must be included in the solution anyway, in order to provide the first approximation values for the coordinates of the model, which are necessary for starting the initial iteration cycle.

The setting up of the observational equations (15) or (19) requires computing the coefficients of the matrix of the unknown parameter corrections and of the residuals. All coefficients are obtained by partial differentiation of the formulas (12). The linearization procedure is accomplished by applying the Taylor series and neglecting second and higher order terms. Therefore, an iterative procedure must be provided in the computation, whereby, the results of each cycle are introduced as approximate values to the following cycle. The iteration is repeated until the solution has converged to a pre-established accuracy level.

We now introduce the following computational auxiliaries:

$$(1) = \frac{x^o - x_p^o}{c^o}$$

$$(2) = \frac{y^o - y_p^o}{c^o}$$

$$-\frac{\partial F_1}{\partial X_j}\frac{q}{c} = a_{X_j} = (3) = (1)\,D - A_1$$

$$-\frac{\partial F_1}{\partial Y_j}\frac{q}{c} = a_{Y_j} = (4) = (1)\,E - B_1$$

$$-\frac{\partial F_1}{\partial Z_j}\frac{q}{c} = a_{Z_j} = (5) = (1)\,F - C_1 \qquad (40)$$

$$-\frac{\partial F_2}{\partial X_j}\frac{q}{c} = b_{X_j} = (6) = (2)\,D - A_2$$

$$-\frac{\partial F_2}{\partial Y_j}\frac{q}{c} = b_{Y_j} = (7) = (2)\,E - B_2 \qquad (40)$$

$$-\frac{\partial F_2}{\partial Z_j}\frac{q}{c} = b_{Z_j} = (8) = (2)\,F - C_2$$

$$(9) = (2)\,B_1 - (1)\,B_2$$

The coefficients of the observational equations (12) are now:

$$\frac{\partial F_1}{\partial \alpha} = A_x = -c^o\left((1)\cdot(9) + (7)\right)$$

$$\frac{\partial F_2}{\partial \alpha} = A_y = -c^o\left((2)\cdot(9) - (4)\right)$$

$$\frac{\partial F_1}{\partial \omega} = B_x = +c^o\left[(1 + (1)^2)\sin\kappa^o - (1)\cdot(2)\cos\kappa^o\right] \qquad (41)$$

$$\frac{\partial F_2}{\partial \omega} = B_y = -c^o\left[(1 + (2)^2)\cos\kappa^o - (1)\cdot(2)\sin\kappa^o\right]$$

$$\frac{\partial F_1}{\partial \kappa} = C_x = -c^o\,(2)$$

$$\frac{\partial F_2}{\partial \kappa} = C_y = +c^o\,(1)$$

$$\frac{\partial F_1}{\partial X_o} = D_x = +\frac{c^o}{q}(3) = -\frac{\partial F_1}{\partial X_j} = -J_x \quad ; \qquad \frac{\partial F_1}{\partial c} = G_x = (1)$$

$$\frac{\partial F_2}{\partial X_o} = D_y = +\frac{c^o}{q}(6) = -\frac{\partial F_2}{\partial X_j} = -J_y \quad ; \qquad \frac{\partial F_2}{\partial c} = G_y = (2)$$

$$\frac{\partial F_1}{\partial Y_o} = E_x = +\frac{c^o}{q}(4) = -\frac{\partial F_1}{\partial Y_j} = -K_x \quad ; \qquad \frac{\partial F_1}{\partial x_p} = H_x = +1$$

$$\frac{\partial F_2}{\partial Y_o} = E_y = + \frac{c^o}{q} (7) = - \frac{\partial F_2}{\partial Y_j} = - K_y \quad ; \qquad \frac{\partial F_2}{\partial x_p} = H_y = \text{does not exist}$$

$$\frac{\partial F_1}{\partial Z_o} = F_x = + \frac{c^o}{q} (5) = - \frac{\partial F_1}{\partial Z_j} = - L_x \quad ; \qquad \frac{\partial F_1}{\partial y_p} = I_x = \text{does not exist} \qquad \begin{matrix}(41)\\ \text{cont'd}\end{matrix}$$

$$\frac{\partial F_2}{\partial Z_o} = F_y = + \frac{c^o}{q} (8) = - \frac{\partial F_2}{\partial Z_j} = - L_y \quad ; \qquad \frac{\partial F_2}{\partial y_p} = I_y = + 1$$

The q, A_1, B_1, C_1, A_2, B_2, C_2, D, E, F terms are computed with formulas (12), (13) and the approximation values of the unknown parameters.

It should be mentioned that, for special cases where the orientations of the various camera axes are such that the approximation values for the rotational parameters can be assumed in all iteration cycles to be either 0, $\frac{\pi}{2}$ or multiples there-of, the differential quotients as given in (41) reduce to well known simple expressions. However, this fact is hardly worth considering for a general solution because the savings in computing time are immaterial if high speed electronic computers are used.

Small and medium sized electronic computers will handle with this method, photogrammetric measuring systems of two and three camera stations with an unlimited number of recorded points. Large computers are adequate to solve the corresponding problem for five and six camera stations. In other words, the suggested method seems feasible to provide a practical analytical solution for photogrammetric measuring problems encountered in terrestrial, laboratory cadastral and ballistic applications. The problems of strip and block triangulation which need to be studied further are considered in a later chapter.

C. On the Use of Celestial Control Points:

The use of celestial targets (mostly fixed stars) is a traditional means in geodesy and photogrammetry for establishing absolute orientation of certain bundles of rays with respect to earth fixed coordinate systems. Terrestrial Photogrammetry has used this method, especially for non-topographical applications. This technique has lately become of interest again in connection with tracking of guided missiles and satellites. In addition, star photography must be considered an excellent means of calibrating precision photogrammetric cameras, because the method not only allows the determination of angular parameters, but the determination of the elements of interior orientation, including distortion coefficients, as well. (See chapter VI of this report).

Whatever the purpose, the method relies simply on the fact that each celestial target point, known by its spherical coordinates, right ascension (α) and declination (δ), provides an absolute control point situated at infinity. Thus, a direction in space is fixed which is not sensitive to changes in the position of the center of projection. Provided that the date and time of exposure, together with the geodetic ellipsoidal coordinates of the center of projection (ϕ and λ) are known, such a direction can be expressed with well-known formulas by a set of azimuth and elevation angles, or as more often used, by so called standard coordinates ξ and η, which are referred to a tangent plane on the unit sphere, described around the center of projection. With proper orientation, the ξ and η coordinates in the notation of this report, are equal to (X), (Y) with (Z) = ± 1. (Compare formulas (12))

The corresponding observational equations are formed exactly like those for absolute control points, with the exception that each of the partial differential coefficients associated with the ΔX_o, ΔY_o, ΔZ_o parameter corrections becomes zero, because of the celestial target points being at infinity.

It is practical to incorporate the influence of astronomical refraction, (see chapter VII of this report) in the computation of the standard coordinates, so that for the corrected rays no further attention need be paid to the refraction problem during the triangulation computations.

Obviously, the sun also is such a celestial target and consequently, the aforementioned facts concerning the data reduction apply to it as well. The use of the sun in aerial photogrammetry is important for strip and block triangulation, and appears to be mandatory for satellite photogrammetry, as well as for special attitude determinations of airborne photogrammetric cameras. The significance of celestial targets is not restricted to the fact that a direction is available which is not affected by accumulated errors. At least equally important is the fact that because of the infinity position of celestial targets, the solution is able to discriminate between rotational and translational parameters. If only terrestrial control points are used these parameters are closely correlated. (Compare [5]).

Generally speaking, we can say that each recorded sun image gives rise to two observational equations in terms of the three rotational parameters (α, ω, κ) of the corresponding station. Obviously, it is necessary that the measured sun image plate coordinates, recorded with a special sun camera, be normalized in such a way that they are compatible with the metric characteristics of the aerial camera taking the corresponding ground photography. Mathematically speaking, it is necessary to normalize the corresponding comparator measurements, by transforming the x,y,c system of the sun camera, with three rotations and three translations, so that it conforms with the x,y, c system of the aerial camera. The corresponding rotational and translational parameters are obtained from preceding camera calibrations, for example by taking star photography.

D. Taking into Consideration Additional Geometric Conditions

In order to complete a generally acceptable analytical solution it is now necessary to study the problem of incorporating certain geometric conditions such as may exist for one or all of the unknown parameters, including both the coordinates of the model and the orientation elements.

In the above outlined solution, a system of normal equations exists, at least temporarily, in terms of both the elements of orientation and the coordinates of the model. The incorporation of any additional conditions existing for any one or all of these unknown parameters, can be performed by a computational technique which was presented by Helmert in [4] dealing in Chapter IV, paragraph 24 on page 196ff with the problem of indirect measurements, the unknown parameters of which have to satisfy certain conditional equations.

Helmert shows in his direct solution, which seems suited for our problem, that it is only necessary to add to the normal equation system the corresponding conditional equations and to restore the quadratic form of the equation system by introducing for each added conditional equation an unknown auxiliary. Assuming the observational equations

$$\begin{aligned} v_1 &= a_1x + b_1y + c_1z - \ell_1 \\ v_2 &= a_2x + b_2y + c_2z - \ell_2 \\ v_3 &= a_3x + b_3y + c_3z - \ell_3 \\ v_4 &= a_4x + b_4y + c_4z - \ell_4 \end{aligned} \tag{42}$$

and the conditional equations

$$\begin{aligned} 0 &= p_o + p_1x + p_2y \\ 0 &= q_o + q_1x + q_3z \end{aligned} \tag{43}$$

We obtain from (42) the corresponding normal equations:

$$\begin{aligned} [aa]\,x + [ab]\,y + [ac]\,z - [a\ell] &= 0 \\ [ab]\,x + [bb]\,y + [bc]\,z - [b\ell] &= 0 \\ [ac]\,x + [cb]\,y + [cc]\,z - [c\ell] &= 0 \end{aligned} \tag{44}$$

This system is being enlarged by adding the conditional equations (43), into the following form:

$$\begin{array}{cccccccl} [aa]\,x & + [ab]\,y & + [ac]\,z & + p_1k_1 & + q_1k_2 & - [a\ell] & = 0 \\ [ab]\,x & + [bb]\,y & + [bc]\,z & + p_2k_1 & \cdot & - [b\ell] & = 0 \\ [ac]\,x & + [cb]\,y & + [cc]\,z & + \cdot & + q_3k_2 & - [c\ell] & = 0 \\ p_1x & + p_2y & \cdot & \cdot & \cdot & + p_o & = 0 \\ q_1x & \cdot & + q_3z & \cdot & \cdot & + q_o & = 0 \end{array} \tag{45}$$

The presence of zeros in some of the diagonal terms must be taken into consideration when the system (45) is being inverted.

All the possible conditional equations which may exist between the unknown parameters of our problem can be divided into two types. In the first group, all conditions shall be considered which introduce additional metric information, as for example, the height of a point above a certain reference surface, or the partial position of a point expressed by latitude ϕ and longitude λ or the slant distance between two relative control points of the model or the differences in flying height between two successive aerial photographs, etc. This type of conditional equation which shall be designated as a metric condition is especially important for the wide field of photogrammetric applications using ground control points.

The use of photogrammetric measuring systems for the determination of trajectories of missiles and the anticipated use of these methods to measure precisely the trajectories of satellites, e.g. for obtaining geodetic basic information, requires the introduction of a second type of conditional equation, where the additional information is not metric in an absolute sense but restricted to the mathematical character of the trajectory. This type of conditional equation shall be designated as trend condition.

(a) Metric Conditional Equations

The mathematical form of a specific metric conditional equation is influenced by the type of metric information given, and by the reference coordinate system introduced for the solution of the specific photogrammetric measuring problem. From the multitude of possible given metric information, the following selection can be assumed to be of special interest for terrestrial control points.

absolute points given by:	Latitude (ϕ), longitude (λ) and elevation (H) with respect to an ellipsoid of revolution, the axes of which are denoted by a and b
partial points given by:	either latitude (ϕ) and longitude (λ), or by elevation (H)
relative points given by:	the slant distance between two relative control points

For computational convenience (see Chapter VII), it is assumed that a local Cartesian system denoted by XYZ is oriented in such a way, that its XY plane is tangent to the reference ellipsoid at the point of local origin, and its +X axis is pointing to the south.

The corresponding metric conditional equations are for:

(1) An absolute control point given by ϕ, λ, H:

$$\begin{aligned} X &= F_1\ (\phi\ \lambda\ H) \\ Y &= F_2\ (\phi\ \lambda\ H) \\ Z &= F_3\ (\phi\ \lambda H) \end{aligned} \tag{46}$$

The system (46) gives the results of a coordinate transformation presented in [7]. No special metric conditional equations become necessary.

(2) A partial control point, given by ϕ and λ:

$$\begin{aligned} &(1)\quad I_\phi X + II_\phi Y + III_\phi Z = \Delta_\phi \\ &(2)\quad I_\lambda X + II_\lambda Y + III_\lambda Z = \Delta_\lambda \end{aligned} \tag{47}$$

and correspondingly, if the approximation values X^o Y^o Z^o satisfy the above conditional equations:

$$\begin{aligned} &(1)\quad I_\phi \Delta X + II_\phi \Delta Y + III_\phi \Delta Z = 0 \\ &(2)\quad I_\lambda \Delta X + II_\lambda \Delta Y + III_\lambda \Delta Z = 0 \end{aligned} \tag{48}$$

where, in formulas (47) and (48):

$$\begin{aligned} I_\phi &= +1 \\ II_\phi &= 0 \\ III_\phi &= \frac{\cos[\phi]\tan\phi - \sin[\phi]\cos\lambda^*}{\sin[\phi]\tan\phi + \cos[\phi]\cos\lambda^*} \qquad \text{with } \lambda^* = \lambda - [\lambda] \\ \Delta_\phi &= X^* + III_\phi Z^* \\ I_\lambda &= \tan\lambda^* \sin[\phi] \\ II_\lambda &= -1 \\ III_\lambda &= \tan\lambda^* \cos[\phi] \\ \Delta_\lambda &= -[X]\tan\lambda^* \end{aligned} \tag{49}$$

$[X]$, $[Y] = 0$, $[Z]$ are the geocentric coordinates of the local origin given by $[\phi]$, $[\lambda]$, $[H] = 0$. The geocentric system is oriented so that its +X axis lies in the meridian plane of the local origin. The $X^* Y^* Z^*$ coordinates are Cartesian coordinates in the local system for the point given by ϕ, λ with $H^* = 0$.

It should be noted that all the coefficients can be computed during the phase of coordinate transformation and consequently enter as constants into the least squares solution of the actual photogrammetric triangulation computations.

(3) A partial control point, given by H:

Such a point is situated on an ellipsoid of revolution, the axes of which are $a^* = (a + H)$ and $b^* = (b + H)$. In order to make the approximation values $X^o\ Y^o\ Z^o$, as they may be obtained with (57), compatible with this condition, it is suggested that first a new Z^o value be computed with:

$$Z^o = \eta_4 + X^o\eta_6 + \eta_7\sqrt{\eta_1\left[1 - \left(\frac{Y^o}{a^*}\right)^2\right] - \left(\frac{X^o + \eta_8}{b^*}\right)^2} \tag{50}$$

The corresponding conditional equation is then

$$I_H\Delta X + II_H\Delta Y + III_H\Delta Z = 0 \tag{51}$$

whereby:

$$\begin{aligned} I_H &= \eta_9 + X^o\ \eta_2 - Z^o\eta_5 \\ II_H &= +Y^o \\ III_H &= \eta_3 + Z^o\eta_1 - X^o\eta_5 \end{aligned} \tag{52}$$

and

$$\begin{aligned} \eta_1 &= \cos^2[\phi] + \left(\frac{a^*}{b_*}\right)^2 \sin^2[\phi] \\ \eta_2 &= \sin^2[\phi] + \left(\frac{a}{b^*}\right)^2 \cos^2[\phi] \\ \eta_3 &= [X]\cos[\phi] + \left(\frac{a^*}{b^*}\right)^2 [Z]\sin[\phi] \\ \eta_4 &= -\frac{\eta_3}{\eta_1} \\ \eta_5 &= \sin[\phi]\cos[\phi]\left\{\left(\frac{a^*}{b^*}\right)^2 - 1\right\} \end{aligned} \tag{53}$$

$$\eta_6 = \frac{\eta_5}{\eta_1} \quad \eta_7 = \frac{a^*}{\eta_1} \quad \eta_8 = [X]\sin[\phi] - [Z]\cos[\phi] \quad \eta_9 = [X]\sin[\phi] - \left(\frac{a^*}{b^*}\right)^2 [Z]\cos[\phi] \quad (53)$$

The η-values are constants and best computed during the preparatory coordinate transformation computations.

(4) In the case that the slant distance (d) between two relative control points i and j is given, we obtain the corresponding metric conditional equation with:

$$d = \left[(X_i - X_j)^2 + (Y_i - Y_j)^2 + (Z_i - Z_j)^2 \right]^{1/2}$$

and correspondingly:

$$(X_i^o - X_j^o)\Delta X_i + (Y_i^o - Y_j^o)\Delta Y_i + (Z_i^o - Z_j^o)\Delta Z_i - (X_i^o - X_j^o)\Delta X_j - (Y_i^o - Y_j^o)\Delta Y_j - (Z_i^o - Z_j^o)\Delta Z_j = \Delta d \quad (54)$$

where:

$$\Delta d = \frac{(X_i^o - X_j^o)^2 + (Y_i^o - Y_j^o)^2 + (Z_i^o - Z_j^o)^2 - d^2}{2}$$

The metric conditional equations (48), (51) and (54) are linear equations in terms of coordinate corrections and have the form of equations given with formulas (43). Consequently, these conditional equations can be used by simply adding the specific equations to the normal equation system as given in formulas (35), which in turn have to be restored to a quadratic form according to the system presented with formulas (45). The resulting system is solved by direct inversion, or solved by the formulation of a system of reduced normal equations by partitioning as described in the previous paragraph. It is obvious that during the process of partitioning and forming the final normal equation system, the minimum size of any one subdivision in the computing process is determined by the specific group of parameter corrections combined by one or more conditional equations. In order to obtain a uniform approach to the computing program, it is possible to consider all points of the model as relative points, introducing the absolutely given control data as metric conditional equations. Considering the given coordinates as approximate values, the corresponding conditional equations with respect to the the three spatial coordinates are $\Delta X_j = 0$, $\Delta Y_j = 0$ and $\Delta Z_j = 0$. These conditional equations, or any selection of them, are added to the corresponding $\mathbf{B}_x^T(\mathbf{A}\mathbf{P}^{-1}\mathbf{A}^T)^{-1}\mathbf{B}_x$ matrix during the process of forming the reduced normal equation system (37). By so doing the coordinate values introduced are enforced. If desired, the corresponding $\mathbf{V}$ residuals can be carried in the $\mathbf{A}$ -matrix as described before.

If on the other hand, the conditional equations are rather simple expressions as e.g. in formulas (47), (48) and (51), it is more practical to use these equations to eliminate certain unknowns in the corresponding observational equations. Thus, the coefficients in the $\mathbf{B_X}$ matrix as given with (41) have to be replaced by:

$$\left.\begin{aligned} (L)_x &= D_x III_\phi - E_x(III_\lambda - I_\lambda III_\phi) - F_x \\ (L)_y &= D_y III_\phi - E_y(III_\lambda - I_\lambda III_\phi) - F_y \end{aligned}\right\} \text{If for points of type 3 the unknowns } \Delta X \text{ and } \Delta Y \text{ are eliminated}$$

$$\text{and } \left.\begin{aligned} (J)_x &= F_x \frac{I_H}{III_H} - D_x, \qquad (K)_x = F_x \frac{II_H}{III_H} - E_x \\ (J)_y &= F_y \frac{I_H}{III_H} - D_y, \qquad (K)_y = F_y \frac{II_H}{III_H} - E_y \end{aligned}\right\} \text{If for points of type 4 the unknown } \Delta Z \text{ is eliminated}$$

In case the systematic errors present in the control data are rather large, it may be desirable to adjust them in such a way that only a specific residual V is obtained for any given control coordinate, independent of the number of rays intersecting at such a point. Such a solution may be obtained by adding to the normal equation system (21), corresponding conditional equations. In general nomenclature, such a conditional equation is, with regard to formulas (15), (16), (34) and (40),

$$V = p^{-1}(ak_1 + bk_2)\frac{q}{c} \tag{55}$$

Thus it is possible to establish for each of the independent combinations of the intersecting rays by pairs, a conditional equation between the two group of associated k_1 and k_2 values, by equalizing the corresponding expressions (55) separately for each given control coordinate. In this way all normal equations associated with a specific point will become interlocked in the $\mathbf{AP^{-1}A^T}$ matrix and the process of stepwise accumulation of the reduced normal equation system (32) or (37) will be rendered much more cumbersome. Numerically speaking, in such cases it appears simpler, and from the theoretical standpoint sufficiently rigorous, to accomplish the adjustment in two separate steps. In the first step a least squares adjustment will be performed according to the presented solution, with the restriction however, that only such absolute given control data are introduced as are necessary for a unique solution. The coordinates of the model thus obtained are then transformed in a second computational step by three translations, three rotations and a scaling factor in such a way that the sum of the squares of the residual distances between the model and all the given control coordinates becomes a minimum. Obviously, with such a method, the photogrammetrically obtained model is being interpolated into the configurations of the given control coordinates. (Compare [3] and [6])

(b) Trend Conditional Equations

Quite similar to the treatment of a metric conditional equation, it is possible to introduce a certain trend condition known to exist between the points to be triangulated, and which is mathematically expressed by a functional relation existing between the coordinates of such points. In a general sense, such conditional equations will resemble metric conditional equations, except that they will not necessarily have any absolutely given parameters. Depending on the mathematical character of a specific trend, the number of points involved in such a trend condition will vary. Correspondingly, the smallest possible subdivision in the process of establishing the final normal equation system by partitioning will be determined by the number of points combined by any one or several trend conditional equations.

E. A Solution for Triangulating Points not Included in the Least Squares Treatment for the Orientation Parameters

Despite the possibility of incorporating any number of relative control points in the general analytical solution for a specific photogrammetric triangulation problem, it sometimes may be desirable to triangulate separately additional points of the model. Consequently, for those points, an independent coordinate determination becomes necessary. The positions of the corresponding rays are determined by the elements of orientation as obtained from an independent preceding least squares adjustment and the corresponding plate measurements. It is obvious that these rays will not intersect due to unavoidable measuring errors. They must be made to intersect so that the sum of the squares of the corrections to be applied to the original plate measurements is a minimum.

It appears advantageous to have an analytical solution which is affected as little as possible by the number of rays involved in a specific triangulation case, and which in addition uses formulas already in use in the computational procedures previously described. Obviously, such a solution is already available with the observational equations (15) if only the coordinate corrections ΔX_j, ΔY_j and ΔZ_j are considered as unknowns. These observational equations lead directly to the corresponding normal equation system formed according to formulas (35) by introducing $(\mathbf{A}\mathbf{P}^{-1}\mathbf{A}^{T})^{-1}$ as $\mathbf{P}$ and $\mathbf{B_0}$ as a null matrix. The necessary coefficients of the observational equations denoted by J_x, J_y, K_x, K_y and L_x and L_y are given in formulas (41).

This approach makes it necessary to compute, as a first step approximation, values for the coordinates of the point under consideration. This may be done efficiently with the use of the formulas (11), which may be written as:

$$
\begin{aligned}
X + \alpha_x Z + \beta_x &= 0 \\
Y + \alpha_y Z + \beta_y &= 0
\end{aligned}
\tag{56}
$$

where:

$$\alpha_x = - \frac{(x-x_p)A_1 + (y-y_p)A_2 + cD}{Q} = - \frac{u}{w}$$

$$\alpha_y = - \frac{(x-x_p)B_1 + (y-y_p)B_2 + cE}{Q} = - \frac{v}{w}$$

$$\beta_x = - (\alpha_x Z_o + X_o)$$

$$\beta_y = - (\alpha_y Z_o + Y_o)$$

The corresponding normal equation system for an n-ray solution is:

$$
\begin{array}{ccl}
\underline{X} & \underline{Y} & \underline{Z} \\
\underline{n} & 0 & +[\alpha_x] + [\beta_x] = 0 \\
 & \underline{n} & +[\alpha_y] + [\beta_y] = 0 \\
 & & +[\alpha\alpha] + [\alpha\beta] = 0
\end{array}
\tag{57}
$$

It should be pointed out that the roots obtained from formula (57) must not be considered as the result of a rigorous least squares solution because there is no indication how nearly this approach minimizes the sum of the squares of the corrections of the original plate measurements. Without doubt the answer will provide an excellent approximation value. As a matter of fact, such an answer may be considered as adequate in itself, if there is evidence of the presence of systematic errors, a situation which renders an additional treatment by a rigorous least squares solution superfluous. Furthermore, the method presented with the formulas (56) and (57) is well suited for computing in each iteration cycle of the least squares solution as described in Chapter IV, the coordinates of the points of the model. (Compare remark Chapter IV (B), page no. (28).

F. The Determination of the Mean Errors of an Observation of Unit Weight of the Elements of the Orientation and of the Triangulation Results

The mean error of an observation of unit weight denoted by m is computed with

$$m = \left(\frac{v^T P v}{r-u}\right)^{1/2} \tag{58}$$

The term $v^T P v$, may be obtained directly from the reduction of the normal equations or by adding the squares of the individual weighted v and V values. The letter r, denotes the number of observational equations and u denotes the number of unknown parameters. Thus the mean error of a specific observation ℓ before adjustment is:

$$m_\ell = \frac{m}{\sqrt{p_\ell}} \tag{59}$$

The computation of m directly from the original measurements, e.g. using the differences of multiple observations, may lead to a value of greater physical significance. The discrepancies between the different values of m, computed with different methods provide means to investigate the presence of systematic errors.

The mean errors of the unknown parameters in a least squares solution are obtained by multiplying m with the corresponding weighting factors. The inverse of the matrix of the coefficients of the final normal equation system, is the matrix of the weighting coefficients. The diagonal elements are the squares of the weighting factors associated with the corresponding unknown parameters.

If the normal equation system as given with formula (31) is directly inverted, the computation of the corresponding mean errors for both the orientation elements and the coordinates of the model does not cause any difficulty. Correspondingly, the mean errors of the unknown parameters associated with the reduced normal equation system in the method of partitioning (x_2 in formula (28) or x_1 in formula (29), respectively) are directly obtained using the diagonal terms of the inverse of the corresponding normal equation matrix. To prove this, we write the matrix of the coefficients of the normal equation system (35) with a notation corresponding to formula (24) as:

$$\begin{array}{c|c} A_{11} & A_{12} \\ \hline A_{12}^T & A_{22} \end{array} \tag{60}$$

The corresponding inverse matrix shall be denoted by:

$$\begin{array}{c|c} Q_{11} & Q_{12} \\ \hline Q_{12}^T & Q_{22} \end{array} \tag{61}$$

According to the definition of the process of inversion, with respect to the notation used in formulas (60) and (61) follows:

$$\begin{bmatrix} A_{11} & A_{12} \\ A_{12}^T & A_{22} \end{bmatrix} \begin{bmatrix} Q_{11} & Q_{12} \\ Q_{12}^T & Q_{22} \end{bmatrix} = \begin{bmatrix} I & O \\ O & I \end{bmatrix} \tag{62}$$

In accordance with the definition of the matrix product, it follows from (62):

$$
\begin{aligned}
&(1) \quad A_{11} Q_{11} + A_{12} Q_{12}^T = I \\
&(2) \quad A_{11} Q_{12} + A_{12} Q_{22} = 0 \\
&(3) \quad A_{12}^T Q_{11} + A_{22} Q_{12}^T = 0 \\
&(4) \quad A_{12}^T Q_{12} + A_{22} Q_{22} = I
\end{aligned}
\tag{63}
$$

With (63) we obtain:

$$Q_{12} = -A_{11}^{-1} A_{12} Q_{22}$$

or

$$
\begin{aligned}
A_{12}^T Q_{12} &= -A_{12}^T A_{11}^{-1} A_{12} Q_{22} \\
+ A_{22} Q_{22} &= + A_{22} Q_{22}
\end{aligned}
\tag{64}
$$

$$A_{12}^T Q_{12} + A_{22} Q_{22} = (63)_4 = I = (A_{22} - A_{12}^T A_{11}^{-1} A_{12}) Q_{22}$$

or

$$Q_{22} = (A_{22} - A_{12}^T A_{11}^{-1} A_{12})^{-1}$$

Formula (64) is identical with the corresponding expression A_2^{*-1} in formula (28), which had to be proven.

In order to compute the mean errors of those parameter corrections, which are eliminated during the process of forming reduced normal equations, according to formulas (37) or (38), the matrices Q_{11} or Q_{22}, respectively, must be obtained. It follows from formulas (63), equations (1) and (2), that

$$Q_{11} = A_{11}^{-1} + A_{11}^{-1} A_{12} Q_{22} A_{12}^T A_{11}^{-1} \tag{65}$$

$$\cancel{Q_{11} = A_{11}^{-1} + A_{11}^{-1} A_{12} Q_{22} A_{12}^{T} A_{11}^{-1}} \tag{65}$$

and correspondingly from the equations (3) and (4)

$$Q_{22} = A_{22}^{-1} + A_{22}^{-1} A_{12}^{T} Q_{11} A_{12} A_{22}^{-1} \tag{66}$$

The computation of the matrices Q_{11} or Q_{22} can not be performed by stepwise accumulation and therefore a considerable computing effort becomes necessary. On the other hand, the relative accuracies within a model are often of pronounced interest. This fact underlines the advantages of a solution which is established either directly on formula (32) or on the reduced normal equation system (38), provided that the corresponding computations can be handled by electronic computers. If it suffices to obtain only the mean errors of the eliminated parameter corrections, it is possible to compute the squares of the corresponding weighting coefficients as the diagonal terms in (65) or (66) by a stepwise accumulation in the same way as the reduced normal equation system was stepwise accumulated.

In case the elements of orientation can be assumed as flawless, the corresponding Q_{22} matrix becomes a null matrix and the A_{11}^{-1} matrix emerges as the weight matrix of the triangulated points. The more excess observations incorporated into the original least squares solution for the orientation parameters, the better this approximation solution will be.

G. An Example of the Described Solution Using Functional Schematics

Figure 5 shows the overlap of 3 photographs which may be considered as being taken either by aerial or ground established cameras. The different types of control points are marked by the following symbols:

- △ : absolute control point, given by X, Y, Z
- ◎ : partial control point, given by X and Y
- ● : partial control point, given by Z
- ⊙ : relative point

The heavily drawn contour line surrounds the area over which a model exists.

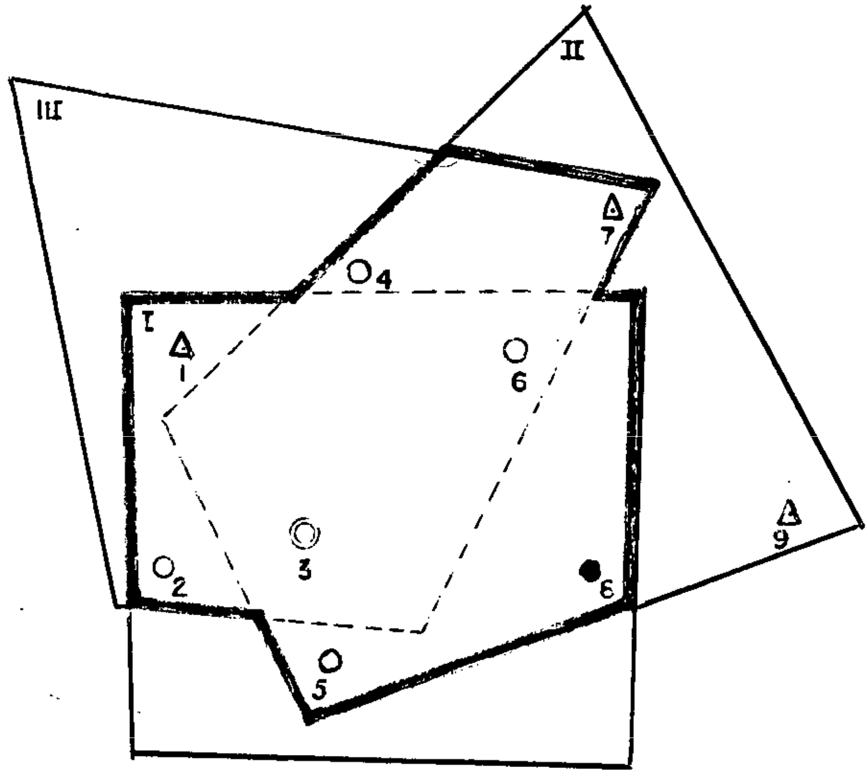


Figure 5

The following tabulation shows the situation interpreted in terms of the recording of the various points (Nos. 1-9) by the various camera stations (Nos. I, II and III)

Point Nos.	Type of Point	Camera Stations at which the point was recorded I	II	III
1	△	x	-	x
2	○	x	-	x
3	◎	x	x	x
4	○	-	x	x
5	○	x	x	-
6	○	x	x	x
7	△	-	x	x
8	●	x	x	-
9	△	-	x	-

Counting the cross marks, we see that 19 individual rays are present. Each ray leading to 2 equations, we have, $19 \times 2 = 38$ observation equations for the 33 unknowns which (assuming for example, the three elements of interior orientation of each camera orientation as known) are composed of $3 \times 6 = 18$ elements of orientation and $(9 \times 3) - 12 = 15$ coordinates of the model. As an interesting by-product, it may be mentioned, that the evaluation of the chosen problem, although slightly overdetermined, would not be possible with conventional restitution equipment and techniques, because there are only a maximum of 4 intersections between corresponding rays of any one pair of photographs, as can be seen from the above tabulation.

With the help of this tabulation we form the corresponding 38 observational equations which are shown schematically in Fig. 6. In Fig. 6 each line represents an observational equation formed according to formulas (18) with the terms given in formulas (15), (40) and (41). The individual **A** matrices and similarly the corresponding **P**-matrices actually are a sequence of completely separated sub-matrices. An inspection of Fig. 6 shows the fact, which has already been mentioned, that in case there are no V's, the **A** matrices become a sequence of unit matrices.

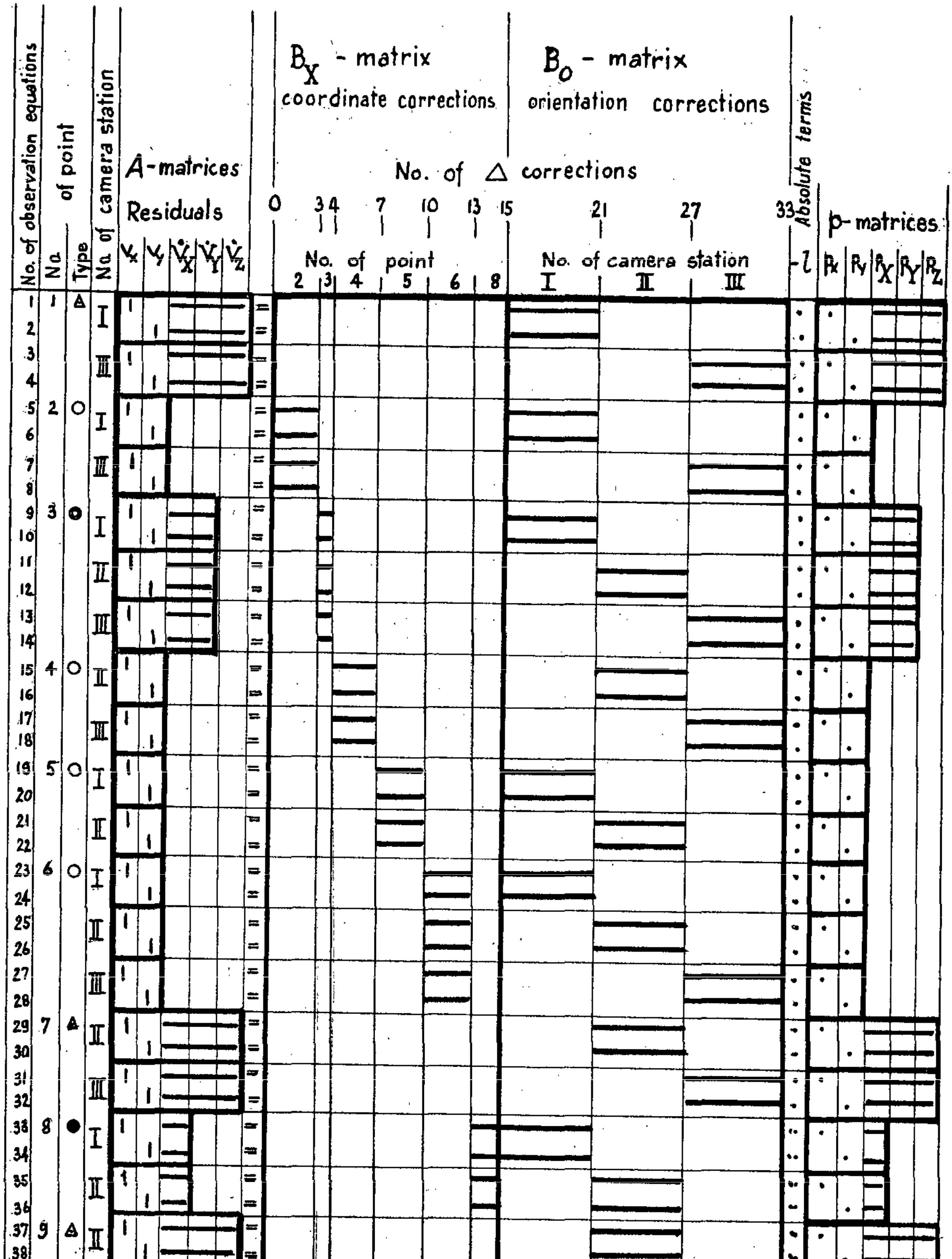
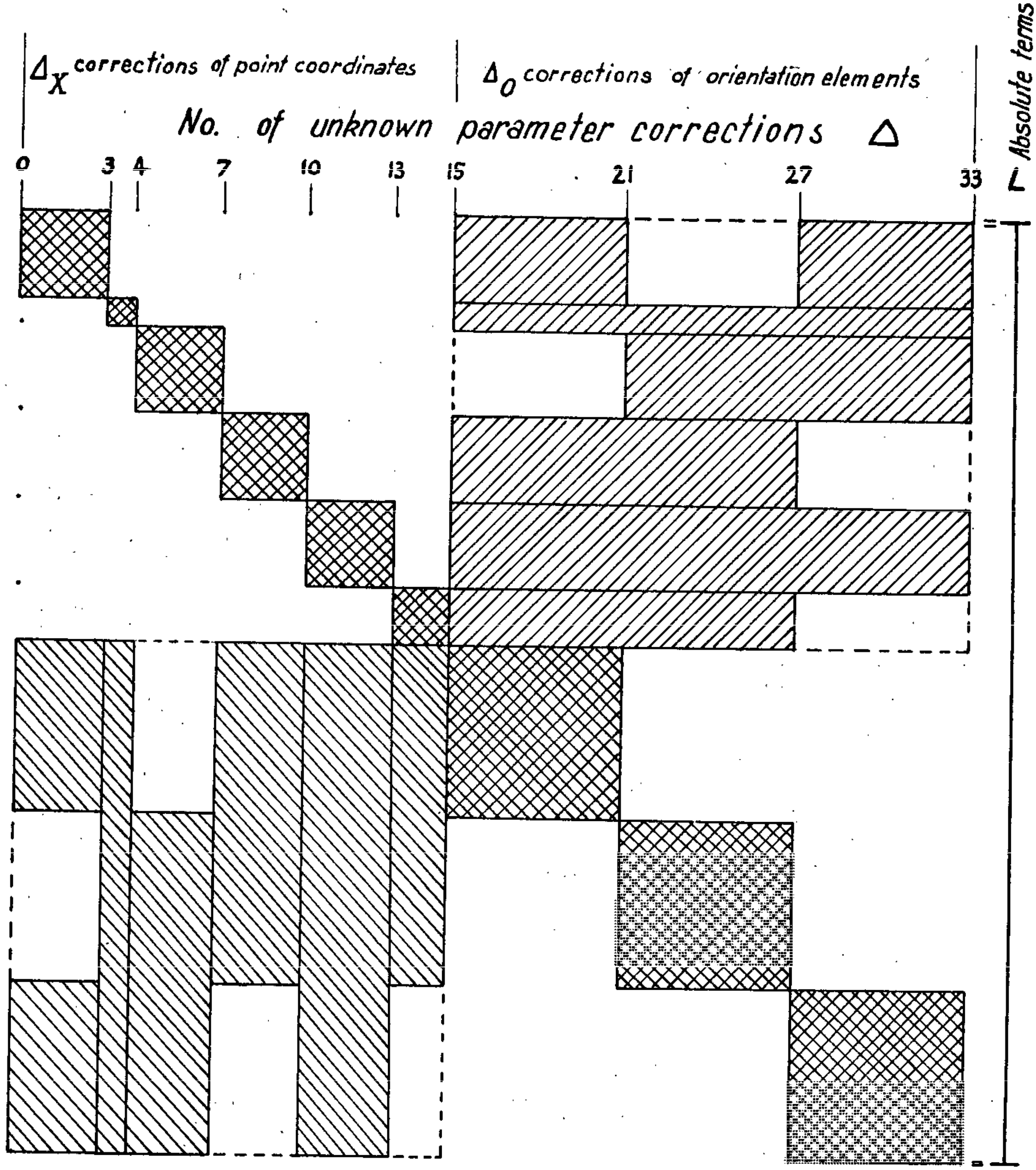

No. of observation equations
No. of point
Type
No. of camera station
A-matrices
Residuals
v_x v_y $\dot{V}_X$ $\dot{V}_Y$ $\dot{V}_Z$
B_X - matrix
coordinate corrections
B_O - matrix
orientation corrections
No. of △ corrections
0 3 4 7 10 13 15 21 27 33
No. of point
2 3 4 5 6 8
No. of camera station
I II III
Absolute terms
-l
p-matrices
p_x p_y p_X p_Y p_Z

Fig. 6



Δ_X corrections of point coordinates
Δ_O corrections of orientation elements
No. of unknown parameter corrections Δ
Absolute terms
L
0
3
4
7
10
13
15
21
27
33

Fig. 7

The actual character of a specific control point with respect to its absolute, partial or relative value is of no concern in setting up the corresponding observational equations. In case any control coordinates are given, it is only necessary to eliminate the corresponding parameter corrections in the B_X-matrix. Therefore, no coordinate corrections appear for the absolute control points Nos. 1, 7 and 9, the partial control point 3 appears only with one, and the partial control point 8 with only two coordinate corrections.

The system of corresponding normal equations formed according to formula (31) is presented in Figure 7.

V. A REFERENCE FOR AN AUXILIARY COORDINATE TRANSFORMATION

The subject of a specific coordinate transformation is not necessarily connected with the subject of this report. However, the application of photogrammetry, especially for geodetic purposes, unavoidably confronts the user with the problem of converting geodetic ellipsoidal coordinates (latitude ϕ, longitude λ, and elevation H, as referred to an ellipsoid of revolution) into a system of arbitrarily oriented Cartesian coordinates and vice versa.

A solution for this problem is given in [7] under the title "Some Remarks on the Problem of Transforming Geodetic Ellipsoidal Coordinates into Cartesian Coordinates with the Help of the Reduced Latitude". Obviously, such a solution includes the establishment of a geocentric Cartesian system, which is but a special case of the general coordinate transformation problem.

VI. DETERMINATION OF RADIAL DISTORTION

Distortion Δ is positive if the image point is displaced away from the principal point.

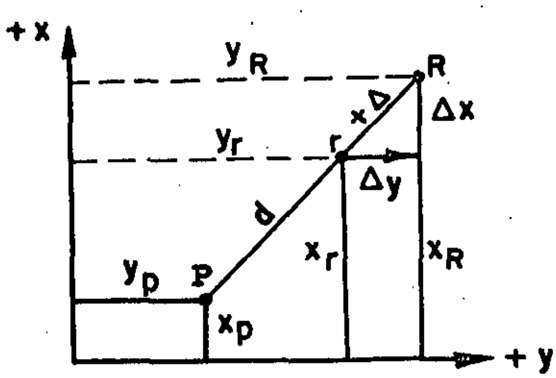


Figure 8

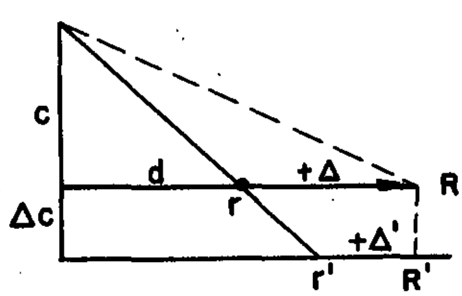


Figure 9

From Fig. 8:

$$d = \left[(x_r - x_p)^2 + (y_r - y_p)^2\right]^{1/2} \tag{67}$$

Assuming that the distortion Δ can be expressed by

$$\Delta = K_o d + K_1 d^3 + K_2 d^5 + K_3 d^7 + \ldots\ldots \tag{68}$$

and

$$\Delta x = \frac{\Delta}{d}(x_r - x_p) = (K_o + K_1 d^2 + K_2 d^4 + K_3 d^6 + \ldots\ldots)(x_r - x_p)$$

$$\Delta y = \frac{\Delta}{d}(y_r - y_p) = (K_o + K_1 d^2 + K_2 d^4 + K_3 d^6 + \ldots\ldots)(y_r - y_p) \tag{69}$$

Due to the character of lens distortion, it is immaterial if d is interpreted as the radial distance with or without distortion.

From Fig. 9:

$$\frac{\Delta - \Delta'}{\Delta c} = \frac{d}{c} \tag{70}$$

or

$$\Delta' = \Delta - \frac{\Delta c}{c} d$$

Substituting formula (68) in (70)

$$\Delta' = K_o' d + K_1 d^3 + K_2 d^5 + K_3 d^7 + \ldots\ldots \tag{71}$$

whereby

$$K_o' = K_o - \frac{\Delta c}{c}$$

K_o' is zero if the condition is introduced that

$$K_o = \frac{\Delta c}{c} \tag{72}$$

Therefore, assuming that in the orientation calibration the principal distance c is considered as unknown, that is to say that the solution is not deprived of the possibility of computing a Δc-correction in each iteration, we may present the remaining distortion by:

$$\Delta' = K_1 d^3 + K_2 d^5 + K_3 d^7 + \dots \tag{73}$$

and correspondingly with formulas (69) and (73) and Fig. 8:

$$\begin{aligned} x_R &= x_r + (K_1 d^2 + K_2 d^4 + K_3 d^6 + \dots)(x_r - x_p) \\ y_R &= y_r + (K_1 d^2 + K_2 d^4 + K_3 d^6 + \dots)(y_r - y_p) \end{aligned} \tag{74}$$

From formulas (12) and with $\begin{cases} x_R = \ell_x + v_x \\ y_R = \ell_y + v_y \end{cases}$ (75)

and formulas (74) the observation equations are:

$$\begin{aligned} \ell_x + v_x &= \frac{cm}{q} (1 + K_1 d^2 + K_2 d^4 + K_3 d^6 + \dots) + x_p = F_1 \\ \ell_y + v_y &= \frac{cn}{q} (1 + K_1 d^2 + K_2 d^4 + K_3 d^6 + \dots) + y_p = F_2 \end{aligned} \tag{76}$$

where

$$d = \frac{c}{q} (m^2 + n^2)^{1/2}$$

$$m = (X)A_1 + (Y)B_1 + (Z)C_1$$

$$n = (X)A_2 + (Y)B_2 + (Z)C_2$$

Considering K_1, K_2 and K_3 as a sufficient number of distortion parameters we obtain from the Taylor expansion for the right-hand side of the above equation according to formulas (15), neglecting terms of second and higher order,

$$\begin{aligned} v_x + a_x \dot{V}\chi + \dots &= \frac{\partial F_1}{\partial \alpha} \Delta\alpha + \dots + \frac{\partial F_1}{\partial K_1} \Delta K_1 + \frac{\partial F_1}{\partial K_2} \Delta K_2 + \frac{\partial F_1}{\partial K_3} \Delta K_3 - \Delta\ell_x \\ v_y + b_x \dot{V}\chi + \dots &= \frac{\partial F_2}{\partial \alpha} \Delta\alpha + \dots + \frac{\partial F_2}{\partial K_1} \Delta K_1 + \frac{\partial F_2}{\partial K_2} \Delta K_2 + \frac{\partial F_2}{\partial K_3} \Delta K_3 - \Delta\ell_y \end{aligned} \tag{77}$$

Because the unknowns K_1, K_2, and K_3 are linear in the observational equations (76), their approximations in each iteration may be taken equal to zero; consequently the coefficients of the remaining unknown orientation parameters remain the same as given by formula (41).

The incorporation of the distortion determination requires merely the addition of the partial differentials with respect to the unknowns K_1, K_2 and K_3 to the system of observational equations.

They are, for the x-equation:

$$
\begin{aligned}
\text{for} \quad & K_1 : M_x = d^2 \cdot \frac{cm}{q} = d^2 \cdot (\ell_x^o - x_p^o) \\
& K_2 : N_x = d^4 \cdot \frac{cm}{q} = d^4 \cdot (\ell_x^o - x_p^o) \\
& K_3 : O_x = d^6 \cdot \frac{cm}{q} = d^6 \cdot (\ell_x^o - x_p^o)
\end{aligned}
$$

and for the y-equation: (78)

$$
\begin{aligned}
\text{for} \quad & K_1 : M_y = d^2 \cdot \frac{cn}{q} = d^2 (\ell_y^o - y_p^o) \\
& K_2 : N_y = d^4 \cdot \frac{cn}{q} = d^4 (\ell_y^o - y_p^o) \\
& K_3 : O_y = d^6 \cdot \frac{cn}{q} = d^6 (\ell_y^o - y_p^o)
\end{aligned}
$$

where

$$d^2 = (\ell_x^o - x_p^o)^2 + (\ell_y^o - y_p^o)^2 = C_y^2 + C_x^2 = C_{xy}$$ (for the meaning of C_y and C_x see formulas (40) and (41))

Consequently we obtain:

$$
\begin{aligned}
M_x &= C_{xy} \cdot C_y & \qquad -M_y &= C_{xy} \cdot C_x \\
N_x &= C_{xy}^2 \cdot C_y & \qquad -N_y &= C_{xy}^2 \cdot C_x \qquad (79) \\
O_x &= C_{xy}^3 \cdot C_y & \qquad -O_y &= C_{xy}^3 \cdot C_x
\end{aligned}
$$

The least squares solution gives, besides the orientation unknowns, the distortion coefficients K_1, K_2 and K_3. The distortion Δ' can now be computed with formula (73). However, usually the distortion curve is presented in such a way that $\Delta = 0$, for a suitably chosen $\hat{d}$. Thus analogous to formula (71), the condition that

$$K_o^* \hat{d} = -(K_1 \hat{d}^3 + K_2 \hat{d}^5 + K_3 \hat{d}^7) \qquad (80)$$

must be satisfied.

From (73) the right side of formula (80) equals $-\Delta'_{\hat{d}}$

$$K_0^* = -\frac{\Delta'_{\hat{d}}}{\hat{d}} = -(K_1 \hat{d}^2 + K_2 \hat{d}^4 + K_3 \hat{d}^6) \qquad (81)$$

and the final distortion curve is now

$$\Delta = K_o^* d + K_1 d^3 + K_2 d^5 + K_3 d^7$$

$$\qquad (82)$$

or

$$\Delta = d \left[K_o^* + K_1 d^2 + K_2 d^4 + K_3 d^6 \right]$$

The "focal length" associated with this distortion curve is again, in analogous to (71),

$$c^* = c(1 - K_o^*) \qquad (83)$$

Accuracy Considerations:

The mean error of an observation of unit weight denoted by m is computed according to (58)

$$m = \left(\frac{v^T P v}{r-u-2k}\right)^{1/2} \qquad (84)$$

where r is the number of observation equations

u is the number of unknown geometrical parameters and

K is the number of unknown distortion parameters carried in the solution.

As described in Chapter IV E. the inverse of the normal equation matrix is the weight matrix of the unknown parameters of the solution and consequently the mean errors of the distortion parameters can be computed directly.

The mean errors of a computed distortion Δ for an arbitrarily chosen d remains to be computed.

$$m_\Delta = m \sqrt{Q_\Delta} \tag{85}$$

Substituting (81) into (82) we obtain

$$\Delta = d \left[K_1(d^2 - \hat{d}^2) + K_2(d^4 - \hat{d}^4) + K_3(d^6 - \hat{d}^6)\right]$$

which may be written as

$$\Delta = d\left[f_1K_1 + f_2K_2 + f_3K_3\right] \tag{86}$$

denoting

$$\mathbf{f} = \begin{bmatrix} f_1 \\ f_2 \\ f_3 \end{bmatrix} \qquad \text{whereby} \qquad \begin{aligned} f_1 &= d(d^2 - \hat{d}^2) \\ f_2 &= d(d^4 - \hat{d}^4) \\ f_3 &= d(d^6 - \hat{d}^6) \end{aligned}$$

we may write (85) as

$$m_\Delta = m\left[\mathbf{f}^T \mathbf{Q}_K \mathbf{f}\right] \tag{87}$$

where $\mathbf{Q}_K$ is the sub-matrix of the inverse of the original normal equation system associated with the unknown distortion parameters K_1, K_2 and K_3.

VII. CONSIDERATION OF REFRACTION

Similar to distortion, refraction causes the tangent to the actual light ray, at the center of projection, to deviate from a line expressing the condition of co-linearity between object point, center of projection and image point.

In order to compensate for refraction, it is necessary to incorporate a corresponding correction in the computation of the x^o and y^o values in formulas (12) and (15), respectively. Figure 10 shows the basic situation, assuming that the local Cartesian system is oriented in such a way that the (X) (Y) plane is normal to the plumb line direction at the center of projection and further assuming that refraction acts only on the elevation angle.

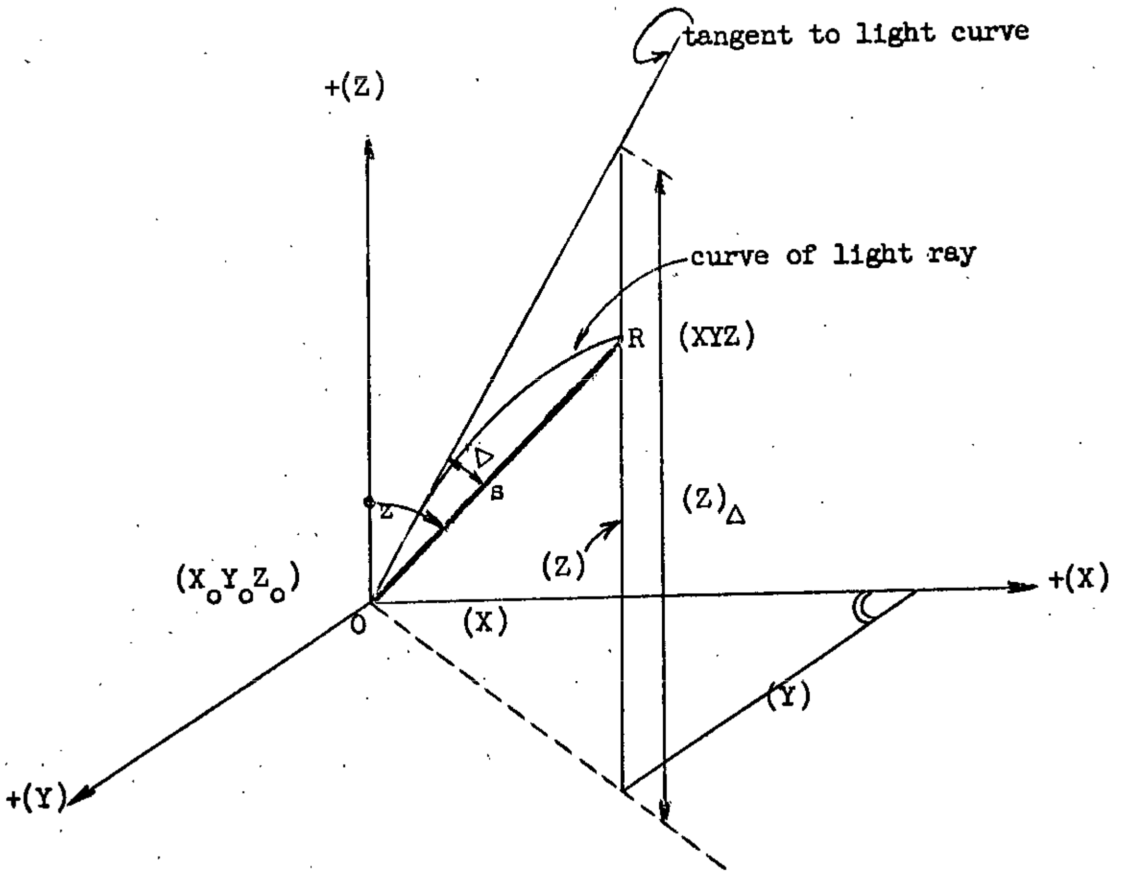
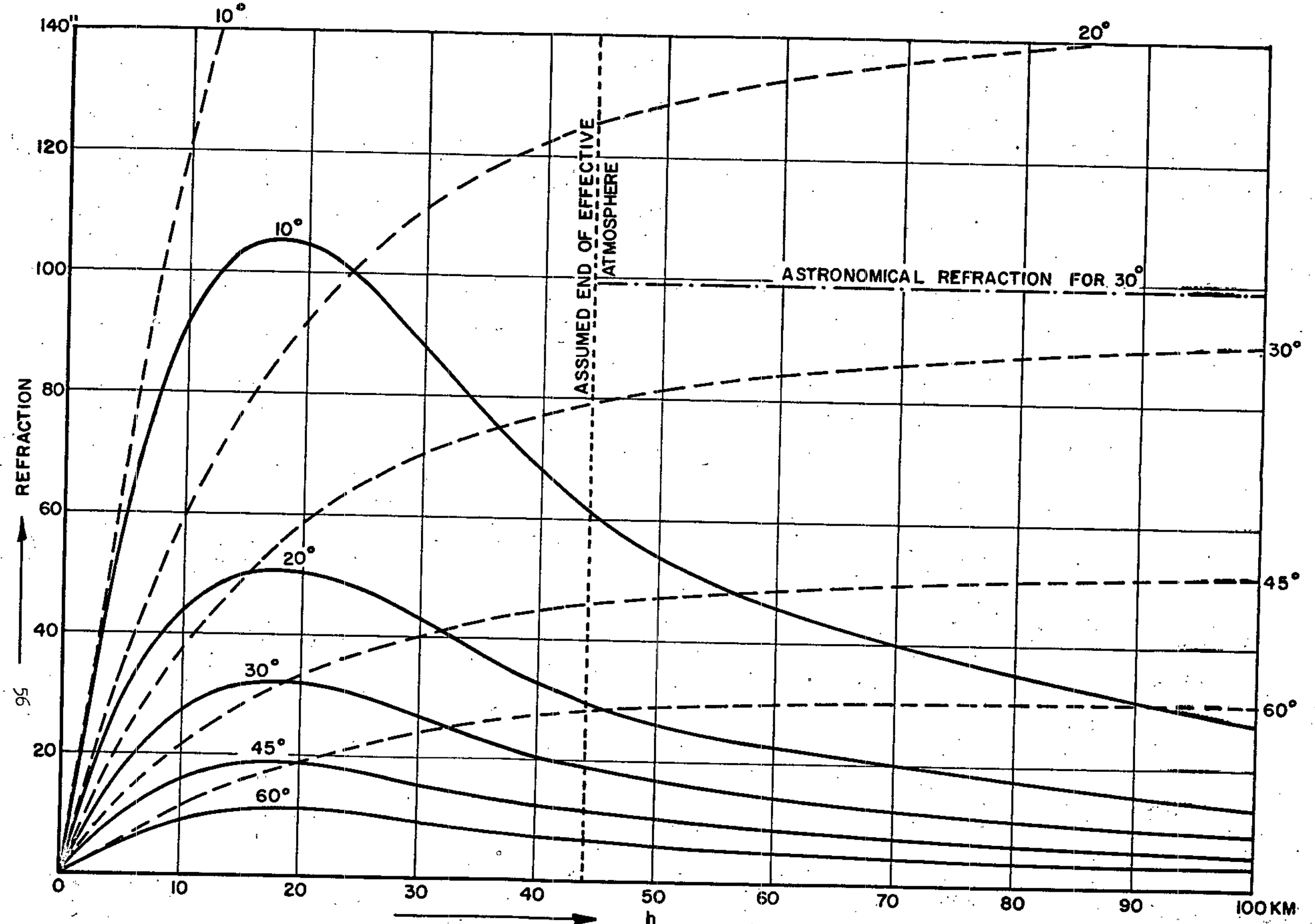


Figure 10



140"
120
100
80
60
40
20
0
REFRACTION
10
20
30
40
50
60
70
80
90
100 KM
h
10°
20°
30°
45°
60°
ASSUMED END OF EFFECTIVE ATMOSPHERE
ASTRONOMICAL REFRACTION FOR 30°

FIGURE 11

We obtain from Fig. (10):

$$\text{ctn } z = \frac{(Z)}{\sqrt{(X)^2 + (Y)^2}} \tag{88}$$

Δ denotes the correction to the zenith angle due to refraction.

$$\Delta = F(X_o\ Y_o\ Z_o,\ s,\ z,\ M) \tag{89}$$

M are meteorological parameters and

$$s = \left[(X)^2 + (Y)^2 + (Z)^2\right]^{1/2}$$

Consequently, we obtain:

$$(Z)_\Delta = \left[(X)^2 + (Y)^2\right]^{1/2} \text{ctn } (z - \Delta) \tag{90}$$

The computation of x^o and y^o with formulas (12) is now carried out, using for each individual ray, the corresponding (X), (Y), and $(Z)_\Delta$ coordinates. As the (X)(Y)(Z) values converge during the iteration cycles to the final answer, so will the corresponding Δ correction converge to the correct refraction value.

Fig. 11 shows the general character of the refraction values as they must be expected by an observer on the ground (dotted lines) and in the air (solid lines).

Refraction in seconds of arc is presented in its functional relation to the elevation or depression angle of the line of sight and to the height above the reference ellipsoid of target point or observer, respectively.

The tables No. 1 and No. 2 show the same information in somewhat more detail. The columns headed by $\Delta/1^o$ show the changes of refraction for 1^o change of a specific elevation or depression angle, respectively.

It appears practical to assume that for aerial precision measurements the 210mm - 60^o lens cone under 20^o tilt presents the most stringent requirements with regard to refraction. Therefore 35^o for a minimum depression angle appears

Refraction
for an observer on the ground

H km \ ε°	10° ″	Δ/1° ″	15° ″	Δ/1° ″	20° ″	Δ/1° ″	25° ″	Δ/1° ″	30° ″	Δ/1° ″	35° ″	Δ/1° ″	40° ″	Δ/1° ″	45° ″	Δ/1° ″	50° ″	Δ/1° ″	55° ″	Δ/1° ″	60° ″	Δ/1° ″	70° ″	Δ/1° ″	80° ″	Δ/1° ″
.5	8	.7	5	.3	4	.2	3	.1	2	.1	2	.0	2	.0	1	.1	1	.0	1	.0	1	.0	0	.0	0	.0
1.0	15	1.4	10	.7	7	.3	6	.2	5	.2	4	.1	3	.1	3	.1	2	.0	2	.1	2	.0	1	.1	0	.1
2.0	30	2.7	19	1.2	14	.7	11	.5	9	.3	7	.2	6	.2	5	.2	4	.2	4	.1	3	.1	2	.1	1	.1
3.0	43	4.1	28	1.9	21	1.1	16	.8	13	.5	11	.4	9	.2	8	.2	6	.2	5	.2	4	.2	3	.2	1	.1
4.0	56	5.2	37	2.4	27	1.4	21	1.0	17	.7	14	.5	12	.4	10	.3	8	.3	7	.2	6	.2	4	.2	2	.1
5.0	69	6.4	45	3.0	33	1.8	26	1.2	21	.9	17	.6	14	.5	12	.4	10	.4	8	.3	7	.3	4	.2	2	.2
6.0	80	7.4	53	3.4	39	2.1	30	1.3	25	1.0	20	.8	17	.6	14	.5	12	.4	10	.3	8	.3	5	.3	2	.3
7.0	92	8.5	60	4.0	44	2.3	35	1.5	28	1.1	23	.9	19	.7	16	.5	14	.4	11	.4	9	.3	6	.3	3	.2
8.0	102	9.5	67	4.4	50	2.6	39	1.7	31	1.2	26	.9	22	.8	18	.6	15	.5	13	.4	10	.4	7	.4	3	.3
9.0	112	10.4	74	4.9	54	2.9	42	1.8	34	1.4	28	1.1	24	.8	20	.7	17	.6	14	.6	11	.4	7	.4	4	.4
10.0	122	11.3	80	5.3	59	3.1	46	2.0	37	1.5	31	1.2	26	.9	22	.8	18	.6	15	.5	12	.5	8	.4	4	.4
12	139	13.0	92	6.0	68	3.5	53	2.4	41	1.6	35	1.3	29	1.1	25	.9	21	.7	17	.6	14	.6	9	.4	4	.4
14	155	15.1	102	6.7	75	3.9	59	2.6	47	1.6	39	1.4	33	1.2	27	.9	23	.8	19	.7	16	.7	10	.5	5	.5
16	169	15.7	111	7.3	82	4.3	64	2.8	52	1.6	43	1.6	36	1.2	30	1.1	25	.9	21	.8	17	.7	11	.6	5	.6
18	182	16.8	120	7.9	88	4.5	69	3.0	55	2.1	46	1.6	38	1.3	32	1.1	27	1.0	22	.8	18	.8	12	.6	6	.5
20	193	17.9	127	8.3	93	4.9	73	3.2	59	2.4	48	1.7	40	1.4	34	1.2	28	1.0	24	.9	20	.8	12	.7	6	.6
25	216	20.0	142	9.3	104	5.4	82	3.6	66	2.6	54	2.0	45	1.6	38	1.3	32	1.1	27	1.0	22	.8	14	.7	7	.7
30	232	21.6	153	10.0	113	5.9	88	3.8	71	2.8	59	2.2	49	1.7	41	1.4	34	1.2	29	1.0	24	1.0	15	.8	7	.7
35	245	22.8	161	10.6	119	6.1	93	4.1	75	2.9	62	2.3	52	1.7	43	1.5	36	1.3	30	1.1	25	1.0	16	.8	8	.8
40	255	23.7	168	11.3	124	6.4	96	4.2	78	3.1	64	2.3	54	1.9	45	1.6	38	1.3	32	1.2	26	1.1	16	.9	8	.8
45	263	24.4	173	11.3	127	6.6	99	4.4	80	3.2	66	2.4	55	1.9	46	1.6	39	1.3	32	1.2	27	1.0	17	.9	8	.9
50	269	24.9	177	11.6	130	6.7	102	4.5	82	3.2	68	2.4	56	1.9	47	1.7	40	1.3	33	1.3	27	1.0	17	.9	8	.9
100	310	27.5	195	12.8	143	7.4	112	4.9	90	3.6	74	2.7	62	2.2	52	1.8	44	1.6	36	1.3	30	1.1	19	1.0	9	.9
200	310	28.8	204	13.4	150	7.7	117	5.1	94	3.7	78	2.8	65	2.2	55	1.9	46	1.6	38	1.4	32	1.3	20	1.1	10	1.0
300	314	29.2	207	13.5	152	7.9	119	5.2	96	3.8	79	2.9	66	2.3	55	1.9	46	1.7	39	1.5	32	1.3	20	1.1	10	1.0
400	316	29.4	208	13.6	153	7.9	120	5.3	97	3.8	80	2.8	66	2.3	56	1.9	47	1.6	39	1.4	32	1.3	20	1.1	10	1.0
500	318	29.5	209	13.7	154	8.0	120	5.3	97	3.8	80	2.9	67	2.3	56	1.9	47	1.6	39	1.4	32	1.3	20	1.1	10	1.0
1000	320	29.8	211	13.8	155	8.0	121	5.3	98	3.8	81	2.9	67	2.3	56	1.9	47	1.7	40	1.5	33	1.3	21	1.1	10	1.1
∞	320	29	211	13.0	156	8.0	122	6.0	99	4.0	82	3.0	68	2.0	57	2.0	48	1.0	40	1.0	33	1.0	21	1.0	10	1.0

Table I

Refraction
for an observer in the air

H km \ $\varepsilon°$	10° ″	Δ/1° ″	15° ″	Δ/1° ″	20° ″	Δ/1° ″	25° ″	Δ/1° ″	30° ″	Δ/1° ″	35° ″	Δ/1° ″	40° ″	Δ/1° ″	45° ″	Δ/1° ″	50° ″	Δ/1° ″	55° ″	Δ/1° ″	60° ″	Δ/1° ″	70° ″	Δ/1° ″	80° ″	Δ/1° ″
.5	8	.7	5	.3	4	.2	3	.2	2	.1	2	.1	2	.1	1	.0	1	.0	1	.0	1	.1	0	.0	0	.0
1.0	15	1.4	10	.6	7	.4	6	.3	4	.2	4	.1	3	.1	3	.1	2	.1	2	.0	2	.1	1	.0	0	.0
2.0	28	2.6	18	1.2	14	.7	11	.4	9	.4	7	.3	6	.2	5	.2	4	.2	4	.2	3	.2	2	.1	1	.1
3.0	40	3.7	26	1.7	19	1.0	15	.7	12	.4	10	.4	8	.3	7	.3	6	.2	5	.1	4	.2	3	.2	1	.1
4.0	51	4.7	33	2.2	25	1.3	19	.8	16	.6	13	.5	11	.4	9	.4	8	.2	6	.3	5	.2	3	.2	2	.2
5.0	60	5.6	40	2.6	29	1.5	23	1.0	18	.7	15	.6	13	.5	11	.3	9	.3	7	.2	6	.2	4	.2	2	.2
6.0	69	6.4	45	2.9	33	1.7	26	1.2	21	.8	17	.6	14	.5	12	.4	10	.4	8	.3	7	.3	4	.2	2	.2
7.0	76	7.1	50	3.3	37	2.0	29	1.3	23	.9	19	.7	16	.6	13	.4	11	.4	9	.4	8	.3	5	.3	2	.3
8.0	82	7.7	54	3.5	40	2.1	31	1.4	25	1.0	21	.8	17	.6	14	.5	12	.4	10	.4	8	.3	5	.3	3	.3
9.0	88	8.1	58	3.8	43	2.2	33	1.5	27	1.1	22	.8	18	.7	16	.5	13	.4	11	.4	9	.4	6	.3	3	.2
10.0	93	8.6	61	4.0	45	2.3	35	1.5	28	1.1	23	.8	19	.6	16	.5	14	.5	11	.4	9	.4	6	.4	3	.3
12	100	9.3	66	4.3	48	2.5	38	1.7	30	1.2	25	.9	21	.7	18	.6	15	.5	12	.5	10	.4	6	.4	3	.3
14	104	10.7	68	4.5	50	2.6	39	1.7	32	1.2	26	1.0	22	.7	18	.6	15	.6	13	.4	11	.4	7	.4	3	.3
16	106	9.9	70	4.6	51	2.7	40	1.8	32	1.3	27	1.0	22	.8	19	.6	16	.6	13	.5	11	.4	7	.4	3	.3
18	106	9.9	70	4.6	52	2.6	40	1.7	32	1.3	27	1.0	22	.8	19	.7	16	.5	13	.4	11	.4	7	.3	3	.3
20	105	9.8	69	4.6	51	2.6	40	1.7	32	1.3	26	.9	22	.7	19	.7	16	.6	13	.5	11	.4	7	.4	3	.4
25	98	9.2	65	4.2	48	2.4	37	1.6	30	1.2	25	.9	21	.7	17	.6	14	.5	12	.4	10	.4	6	.3	3	.4
30	88	8.1	58	3.8	43	2.2	33	1.4	27	1.1	22	.8	18	.7	16	.5	13	.4	11	.4	9	.4	6	.3	3	.2
35	77	7.2	51	3.3	38	2.0	29	1.3	24	.9	20	.7	16	.6	14	.4	11	.4	10	.4	8	.3	5	.3	2	.2
40	68	6.3	45	2.9	33	1.7	26	1.1	21	.8	17	.6	14	.5	12	.4	10	.4	8	.3	7	.2	4	.3	2	.2
45	60	5.6	40	2.6	29	1.5	23	1.0	18	.7	15	.5	13	.4	11	.4	9	.4	8	.3	6	.3	4	.2	2	.2
50	54	5.1	36	2.3	26	1.4	21	.9	17	.6	14	.5	11	.4	10	.3	8	.3	7	.2	6	.2	4	.2	2	.2
100	27	2.5	18	1.1	13	.7	10	.5	8	.3	7	.3	6	.2	5	.2	4	.1	3	.2	3	.1	2	.0	1	.0
200	14	1.2	9	.6	7	.3	5	.3	4	.2	3	.1	3	.1	2	.1	2	.1	2	.1	1	.1	1	.1	0	.0
300	9	.9	6	.4	4	.2	3	.1	3	.1	2	.1	2	.1	2	.1	1	.0	1	.0	1	.0	1	.0	0	.0
400	7	.6	4	.3	3	.2	3	.1	2	.1	2	.0	1	.0	1	.0	1	.0	1	.0	1	.0	0	.0	0	.0
500	5	.5	4	.2	3	.1	2	.1	2	.1	1	.1	1	.0	1	.1	1	.0	1	.1	1	.1	0	.0	0	.0
1000	3	.2	2	.1	1	.0	1	.1	1	.0	1	.0	1	.0	0	.0	0	.0	0	.0	0	.0	0	.0	0	.0
∞	0	—	0	—	0	—	0	—	0	—	0	—	0	—	0	—	0	—	0	—	0	—	0	—	0	—

Table II

to be a practical limit. Table No. 2 shows that for an aerial observer both the absolute amount of refraction and its differential change have their maximum values for a flying height of about 18 km. For a depression angle of 35° the corresponding values are 27" and 1", respectively. Based on practical experience with metric aerial photography, one should be satisfied with a computational accuracy of $\pm 1''$ for the refraction correction. Consequently, the corresponding depression angles for the individual rays can be computed by simple formulas depending on a local Cartesian coordinate system, provided that its origin is within $\pm 1^{\circ} = \sim \pm 100$ km from the camera position. This approach will do justice to strips of 200 km length or blocks with sides of about 150 km.

At the same time it is desirable to minimize the computational difficulties connected with the introduction of certain metric conditions associated with partial control points. A corresponding orientation of the local Cartesian system has been described in Chapter IV - (D), page 35.

Formulas describing refraction corrections have been derived by several authors. In [8] a treatment is given for astronomical refraction including zenith distances $> 90^{\circ}$. In [9] expressions for target points inside the atmosphere have been derived. References with respect to corresponding basic assumptions and formulas may be found in the recently published report, [10]. A summary of the problem of refraction in photogrammetry is found in [11].

M, the meteorological constants referred to in formula (89) are either obtained in accordance with some model of the atmosphere or are determined directly from independent meteorological measurements, or obtained indirectly by additional elevation angle measurements with respect to known points; a procedure which under certain conditions can be made a part of the actual photogrammetric triangulation measurements. If aerial analytical triangulation demands specific meteorological parameters in the area of the flight, it is feasible to eject periodically from the airplane probes for measuring densities and/or temperatures; the corresponding results being recorded by radio link in the airplane.

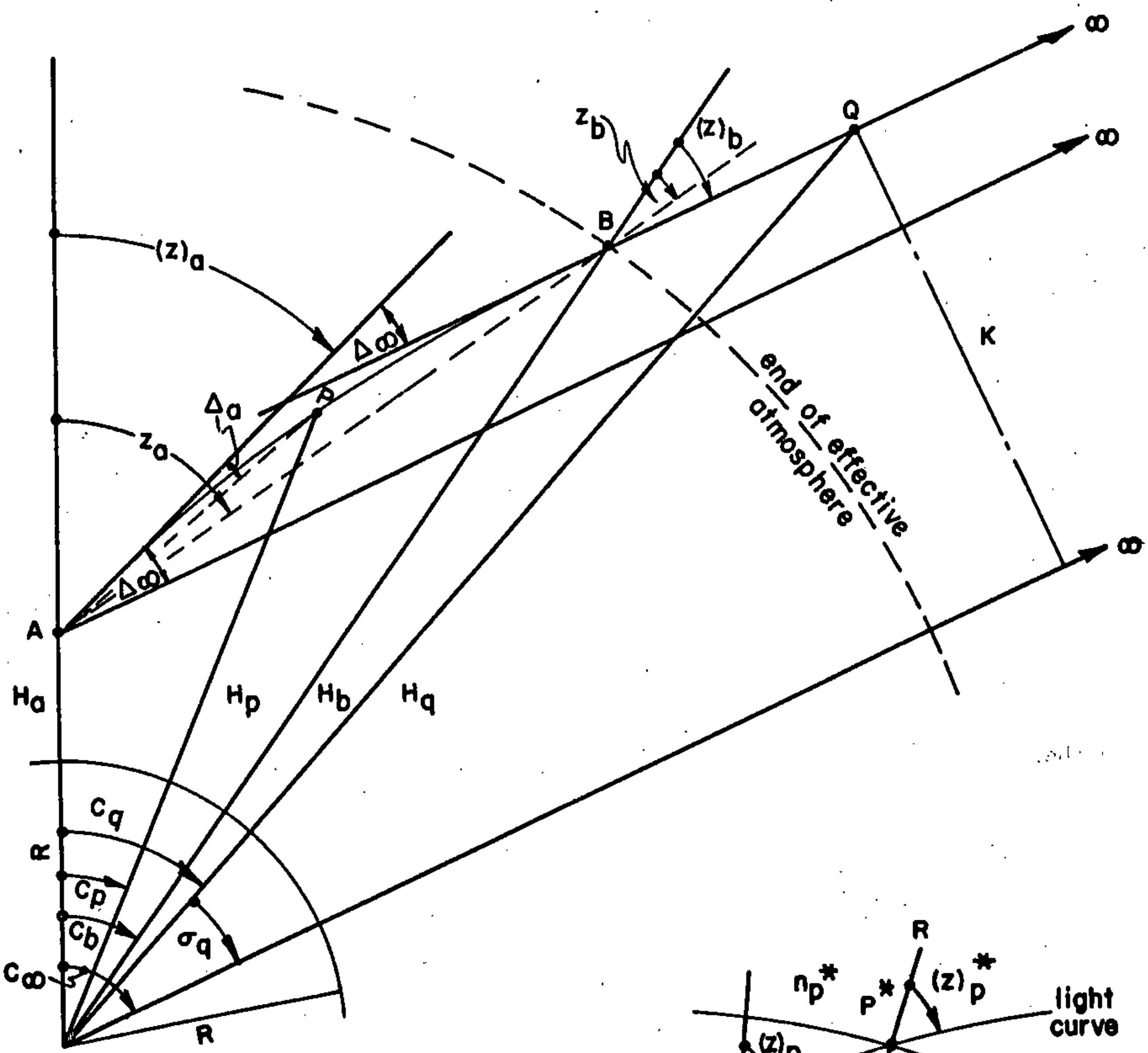


FIGURE 12

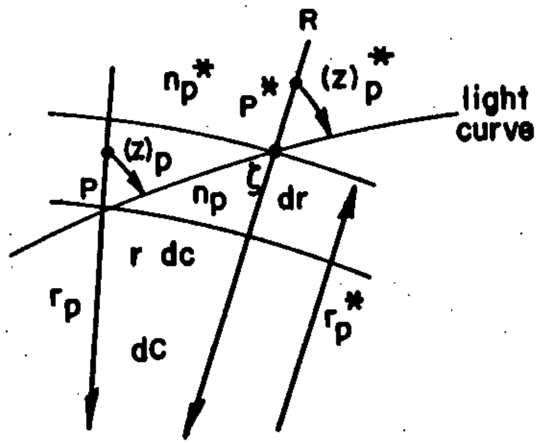


FIGURE (13)

In accordance with the conventional approach we derive from Fig. (13) and Snell's law:

$$\frac{\sin\,(z)_p^*}{\sin\,\zeta} = \frac{n_p}{n_p^*} \qquad n = \text{index of refraction} \tag{91}$$

and from the sine law:

$$\frac{\sin\,(z)_p}{\sin\,\zeta} = \frac{r_p^*}{r_p} \qquad \text{where } r_p = R + H_p \quad \text{(Fig. 12)} \tag{92}$$

with (91) and (92) it follows that:

$$n_p^* \, r_p^* \sin\,(z)_p^* = n_p r_p \sin\,(z)_p = K \tag{93}$$

in particular for point A, $K = n_a r_a \sin\,(z)_a$ where $r_a = R + H_a$

Further from Figure (13), omitting subscripts,

$$dC = \frac{K\;dr}{r\left[(nr)^2 - K^2\right]^{1/2}}, \text{ and, } C_p = K\int_{r_a}^{r_p} \frac{dr}{r\sqrt{(nr)^2 - K^2}} \tag{94}$$

Applying formula (94) to points (Q) outside of the effective atmosphere, where n = unity, we obtain with $r_q = R + H_q$

$$\sigma_q = K\int_{r_q}^{r=\infty} \frac{dr}{r\sqrt{r^2 - K^2}} = \sin^{-1}\frac{K}{r_q} \tag{95}$$

and, correspondingly, from Fig. (12) for all points for which $r_q > r_b$

$$C_q = (z)_a + \Delta_\infty - \sin^{-1}\frac{K}{r_q} \tag{96}$$

From (96) it follows that, for target points situated outside the effective atmosphere, refraction can be expressed in terms of astronomical refraction denoted by Δ_{∞}. According to [8] we have

$$\Delta_{\infty} = \alpha_1 \tan z^* + \alpha_2 \tan^3 z^* + \alpha_3 \tan^5 z^* + \alpha_4 \tan^7 z^* + \ldots \tag{97}$$

with $\operatorname{ctn} 2z^* = \nu_a \operatorname{ctn} (z)_a$ and for air $\nu_a = 8.1578 \sqrt{\frac{273}{T_a}}$

The coefficients $\alpha_1 \ldots \alpha_4$ depend upon the structure of the atmosphere. As outlined in [11] for precision work it will be necessary to compute n_A with the Cauchy equation as function of the effective wave length and obtain an index of refraction profile from Rawinsonde observations. It is then possible with formula (94) to determine for various (z) values the corresponding C_{∞}-values by numerical integration. The corresponding Δ_{∞} values, computed from the relation $C_{\infty} = (z) + \Delta_{\infty}$ (Fig. 12) are now used to compute the specific α-coefficients by fitting the expansion as given by (97) to the computed Δ_{∞} values.

In order to apply formulas (94) or (96) to the case of an aerial observer, it is necessary to establish the relation between $(z)_p$ and $(z)_a$ which is obtained from (93).

$$\sin (z)_a = \frac{K}{n_a r_a} = \frac{n_p r_p \sin (z)_p}{n_a r_a} \tag{98}$$

The true zenith distances z_a and z_p are computed, according to Fig. 12 from

$$\tan z_a = \frac{\sin C_p}{\cos C_p - \frac{r_a}{r_p}} \tag{99}$$

and

$$\tan z_p = \frac{-\sin C_p}{\cos C_p - \frac{r_p}{r_a}} \tag{100}$$

From Fig. 12 we see that for all points outside the effective atmosphere, the refraction as encountered by an observer on the ground and in the air for a specific ray add up to astronomical refraction.

This fact is illustrated in Fig. 11 where, as example, for an elevation and depression angle of 30°, the sums of the ordinates of the solid and dotted lines have been plotted for altitudes outside of the effective atmosphere. The resulting line, designated "astronomical refraction for 30°", agrees with the corresponding value of 99" in Table No. 1 for $\epsilon = 30^\circ$ and $H = \infty$.

For precision triangulation of points within the atmosphere, especially if large distances between observer and target are encountered as in guided missile applications, the numerical integration by formula (94) based on an index of refraction profile will be unavoidable. For many cases, however, especially in aerial photogrammetry, an expression for the refraction can be used which avoids the cumbersome numerical integration.

Assuming a constant atmospheric temperature gradient, refraction expressions were derived in [9] for both an observer on the ground and in the air. However, these formulas are restricted to a situation where one of the end points of the light curve is situated at the height of the ellipsoid of reference. Moreover, the expressions obtained in [9] are unnecessarily complicated for numerical evaluation. In the following a derivation is given which, following the general approach of [9], overcomes the above mentioned restriction and results in expressions more suited for numerical evaluation.

From the well known pressure altitude relationship it follows that for a linear decrease of temperature with altitude,

$$\frac{p}{p_o} = \left(\frac{T}{T_o}\right)^a \quad \text{where } a = -\frac{1}{RL} \quad \text{and } T = T_o + LH \tag{101}$$

R is the gas constant for air, T absolute temperature

and $L = \frac{-dT}{dH}$ the temperature gradient.

Similarly well-known is the density - altitude - pressure relationship.

$$\frac{\rho}{\rho_o} = \frac{p}{p_o}\,\frac{T_o}{T} = \left(\frac{T}{T_o}\right)^{a-1} \tag{102}$$

Finally we introduce the well known relationship between the indexes of refraction and density (compare e.g. [12]). $n^2 - 1 = c\rho$, where c is a constant. With $n = 1 + \alpha$, where α is usually < 0.0003, we may write:

$$\frac{n-1}{n_o-1} = \frac{\alpha}{\alpha_o} = \frac{\rho}{\rho_o} \text{ or } n = 1 + \alpha_o\left(\frac{T}{T_o}\right)^{a-1} \tag{103}$$

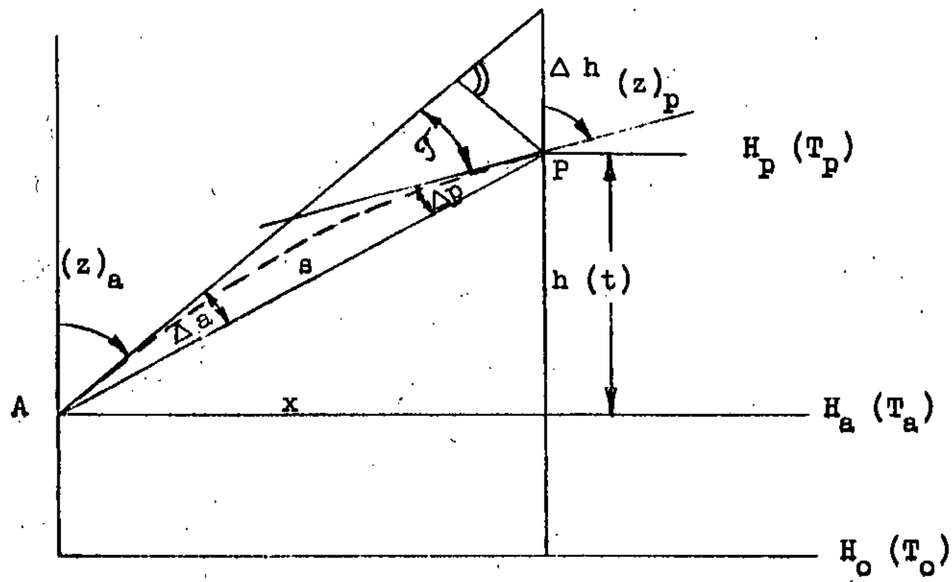


Figure 14

$$H_p - H_a = h = \frac{T_p - T_a}{L} = \frac{t}{L} \tag{104}$$

From Fig. 14 : $x = (h + \Delta h)\tan(z)_a$ (105)

and $$\Delta_a = \frac{\Delta h \sin(z)_a}{s} \approx \frac{\Delta h \sin(z)_a \cos(z)_a}{h} \tag{106}$$

In [13] an expression for x is published, which may be written as:

$$x = h\tan(z)_a + \frac{\tan(z)_a}{\cos^2(z)_a} \int_{H_a}^{H_p} \frac{n_a - n_p}{n_p} dH \tag{107}$$

For optical frequencies n_p is usually < 1.0003 but > 1 and we may therefore write for (107)

$$x = h\tan(z)_a + \frac{\tan(z)_a}{\cos^2(z)_a} \int_{H_a}^{H_p} (n_a - n_p) dH \tag{108}$$

By comparing (105) with (108) it follows from (106):

$$\Delta_a = \frac{\tan(z)_a}{h} \int_{H_a}^{H_p} (n_a - n_p)\, dH \tag{109}$$

Due to the linear relationship between temperature and height, (101), we may write with the notation in Figure 14 and formula (104) :

$$\Delta_a = \frac{\tan(z)_a}{t} \int_{T_a}^{T_p} (n_a - n_p)\, dT \tag{110}$$

From (103):

$$n_a = 1 + \alpha_o \left(\frac{T_a}{T_o}\right)^{a-1}$$

$$n_p = 1 + \alpha_o \left(\frac{T_p}{T_o}\right)^{a-1}$$

$$(n_a - n_p) = \frac{\alpha_o}{T_o^{a-1}} (T_a^{a-1} - T_p^{a-1}) = \frac{\alpha_o T_a^{a-1}}{T_o^{a-1}} \left[1 - \left(1 + \frac{t}{T_a}\right)^{a-1}\right] \tag{111}$$

Substituting (111) into (110) we have:

$$\Delta_a = \frac{\tan (z)_a}{t} \; \frac{\alpha_o \, T_a^{a-1}}{T_o^{a-1}} \int_0^t \left[1 - \left(1 + \frac{t}{T_a}\right)^{a-1} \right] dt \tag{112}$$

$$\Delta_a = \frac{\tan (z)_a}{t} \; \frac{\alpha_o \, T_a^{a-1}}{T_o^{a-1}} \left[t - \frac{T_a}{a} \left(1 + \frac{t}{T_a}\right)^{a} + \frac{T_a}{a} \right] \tag{113}$$

$$\Delta_a = \frac{\alpha_o}{T_o^{a-1}} \; \tan (z)_a \left[T_a^{a-1} - \frac{T_a^a - T_p^a}{a(T_a - T_p)} \right] \tag{114}$$

or

$$\Delta''_a = A \tan (z)_a \left[T_a^{a-1} - \frac{T_a^a - T_p^a}{a(T_a - T_p)} \right] \tag{115}$$

where

$$A = \frac{\alpha_o}{T_o^{a-1}} \rho'' \qquad T_a = T_o + LH_a$$

$$a = -\frac{1}{RL} \qquad T_p = T_o + LH_p$$

From Figure 12 or 14 we have:

$$\tau = \Delta_a + \Delta_p \tag{116}$$

where τ is angular refraction, and Δ_p is the refraction of an aerial observer.

From 93 follows:

$$\frac{n_a}{n_p} = \frac{r_p \sin (z)_a}{r_a \sin (z)_p} \tag{117}$$

with (116), setting $\frac{r_p}{r_a} \approx 1$, and with τ being a small angle,

we may write (117) as:

$$\frac{n_a}{n_p} = 1 + \tau \operatorname{ctn} (z)_a \tag{118}$$

Neglecting again n_p in the denominator, we have:

$$(n_a - n_p) = \tau \text{ ctn } (z)_a \tag{119}$$

or with (111),

$$\tau = \frac{\alpha_o}{T_o^{a-1}} \tan (z)_a \, (T_a^{a-1} - T_p^{a-1}) \tag{120}$$

and with the notation of formula (115),

$$\tau'' = A \tan (z)_a \, (T_a^{a-1} - T_p^{a-1}) \tag{121}$$

The refraction for an aerial observer follows, from (116):

$$\Delta_p'' = A \tan (z)_a \left[\frac{T_a^a - T_p^a}{a(T_a - T_p)} - T_p^{a-1} \right] \tag{122}$$

As mentioned on page 64, τ becomes astronomical refraction (Δ_∞) for all points outside the effective atmosphere for which $T_p = 0$. Consequently (121) reduces for $T_p = 0$ to:

$$\Delta_\infty'' = A \tan (z)_a \, T_a^{a-1} \tag{123}$$

Introducing (123) into (115) we obtain:

$$\Delta_a'' = \Delta_\infty'' - A \tan (z)_a \frac{T_a^a - T_p^a}{a(T_a - T_p)} \tag{124}$$

$$\Delta_a'' = \Delta_\infty'' \left[1 - \frac{T_a^a - T_p^a}{a T_a^{a-1} (T_a - T_p)} \right] \tag{125}$$

whereby for precision work Δ_∞'' can be computed from (97).

Refraction can now be taken into consideration by computing either Δ or $(z - \Delta)$ as required in formula (90).

Type of application	Location of observer or target point respectively. Outside effective atmosphere	Inside effective atmosphere
Aerial Photogrammetry	Formulas (100), (96) (97) and (93)	Formula (122)
Terrestrial Photogrammetry	Formulas (99), (96), (97) and (93)	Formula (115) or (124) and (97) or (99) & (94) in order of increasing precision

Attention must be given to the fact that the computation of z for a specific ray is carried out with the coordinates of the center of projection and of a specific control point. Thus z, and not (z), is being obtained (Fig. 12). In extreme cases of precision terrestrial photogrammetry, it may therefore become necessary to compute a series of refraction corrections according to the following steps:

1. $\Delta' = f\, z$
2. $(z)' = z - \Delta'$
3. $\Delta'' = f(z)'$ etc.

The computations are continued until the specific Δ-value stabilizes.

Provided $(z)_a + (z)_p$ is sufficiently close to 180° (flat earth geometry), formulas (115) and (122) can be combined in an expression valid for both a ground based or airborne camera. Introducing the notation of this report, the center of projection by 0 and the target point by R, we obtain:

$$\Delta'' = A \tan (z)_0 \left[T_0^{a-1} - \frac{T_0^a - T_R^a}{a(T_0 - T_R)} \right] \qquad (126)$$

VIII. THE TREATMENT OF STRIP AND BLOCK TRIANGULATION

It has been shown that the matrix of normal equations (35), illustrated in Fig. 15, expresses the general photogrammetric problem. The contents of that matrix include the solution for the triangulation of an unlimited block and obviously, as a special case, the triangulation of an unlimited strip.

Such matrices resemble each other in the geometric arrangement because successive groups of ground points are being photographed from a certain number of successive camera stations. This fact is reflected in the arrangement of the coefficients in the $B_X^T(AP^{-1}A^T)^{-1}B_O$ matrix and its transpose. These matrices are only partially filled, forming an escalator pattern as shown in Fig. 15.

The example is a strip with 2/3 overlap, where it is assumed that six points are located in each trilap-area on the ground (I, II .....N). The corresponding reduced normal equation system (37) itself becomes a symmetrical escalator matrix, grouped along the main diagonal as shown in Fig. 16.

Fig. 17 shows a reduced normal equation matrix for the determination of the corresponding Δ_X vectors of a block of 7 x 7 photographs flown with 2/3 longitudinal and 2/3 lateral overlap. In this way, each portion of the ground is being recorded on nine photographs. In the example it was assumed that one point is located in each one of the nine times covered ground sections. The matrix as shown in Fig. 17 is typical in its arrangement for any block triangulation.

The size of the cross shaped openings which are filled with zeros obviously increases with an increase in the length of the sides of the block under consideration.

If one uses the aforementioned schematic of the general solution, present day electronic computers allow, without undue difficulties, the formation of the reduced normal equation systems as presented in Figs. 16 and 17. However, the problem remains to invert these normal equation systems. Generally speaking, both strip and block triangulation will result in normal equation systems with too many unknowns for direct inversion.

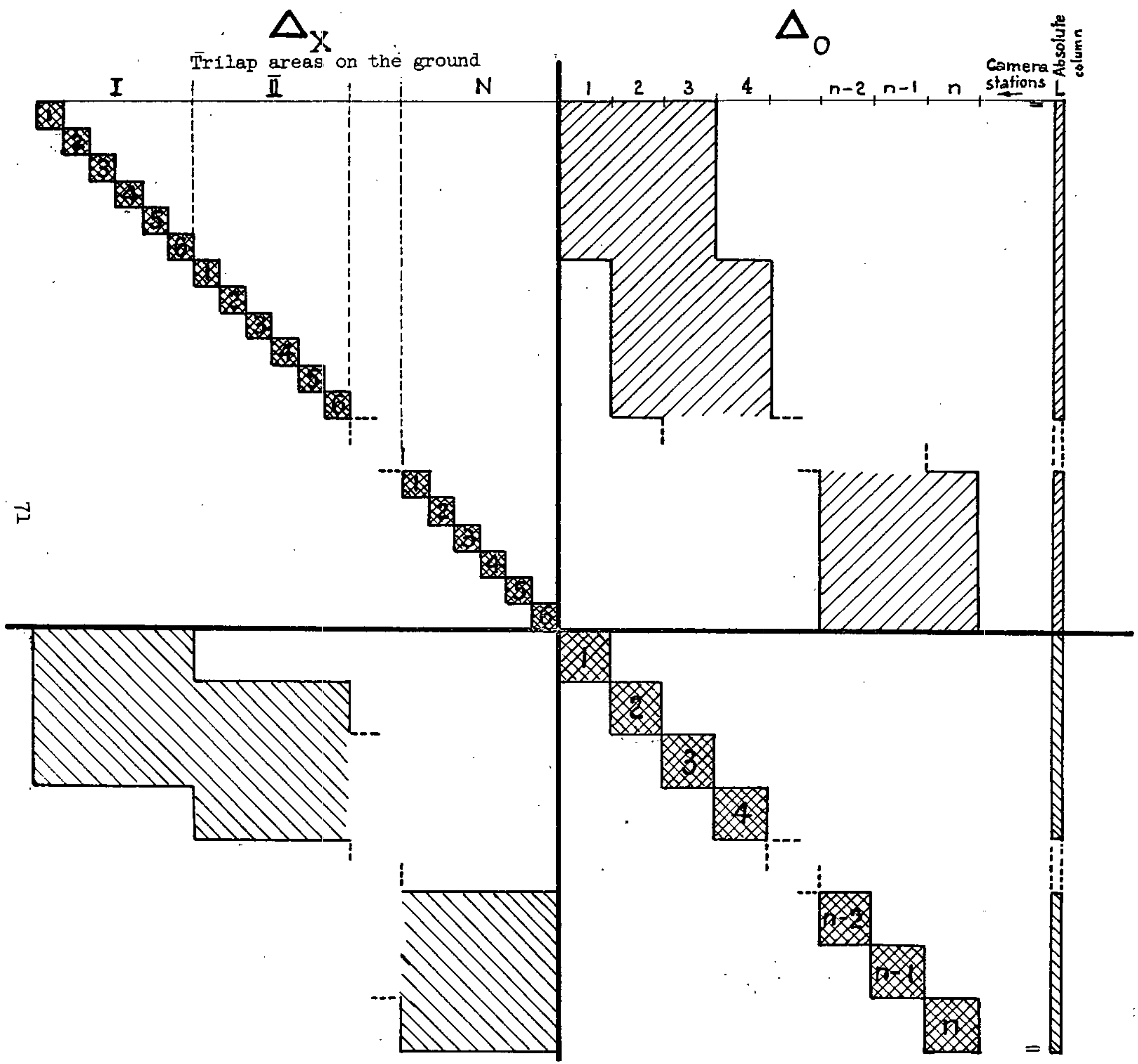
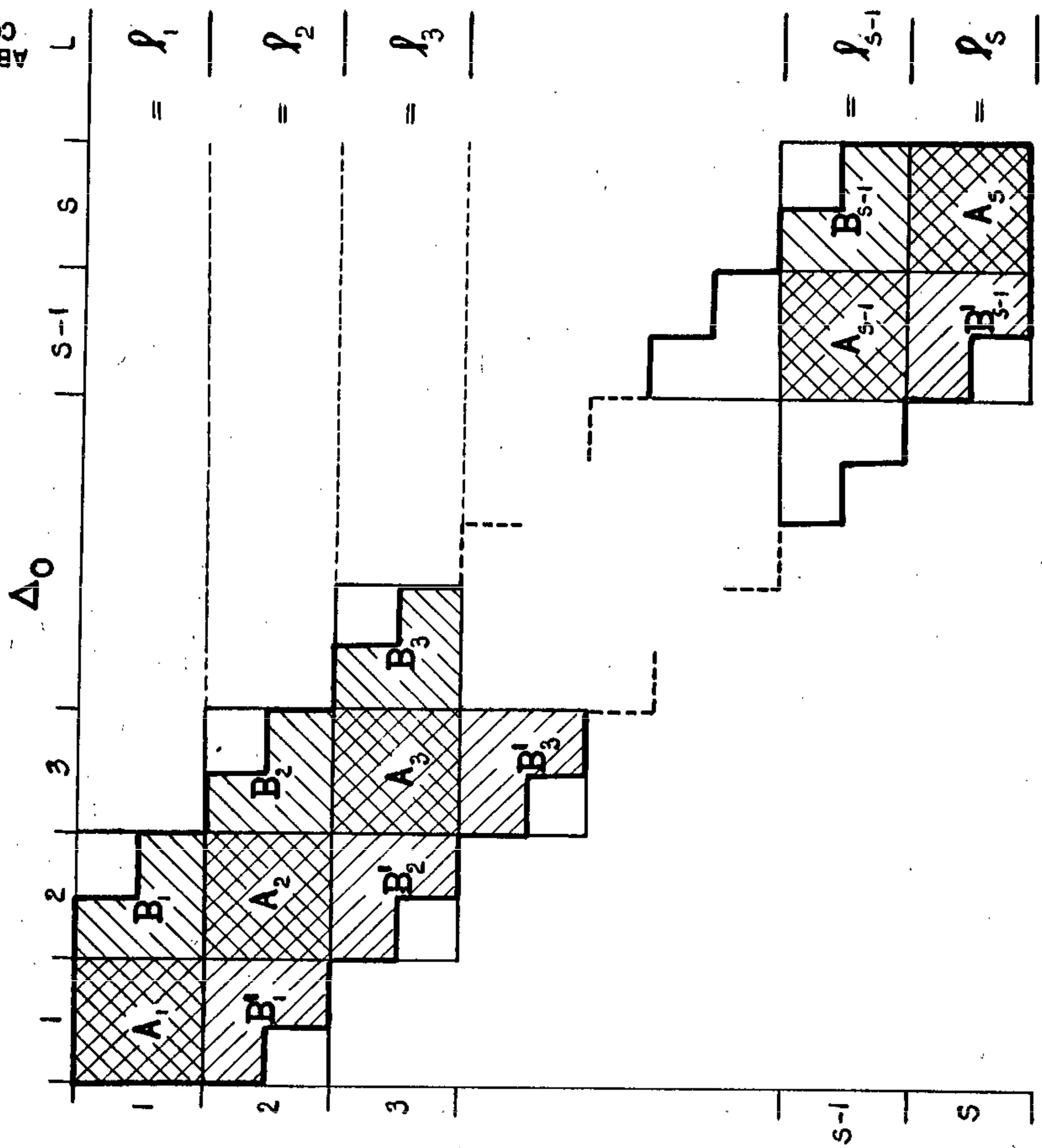

Δ_X
Trilap areas on the ground
I
II
N
Δ_O
1
2
3
4
n-2
n-1
n
Camera stations
Absolute column
L

Figure 15



ABSOLUTE COLUMN
L
Δ_0
1
2
3
s-1
s
A_1
B_1
B'_1
A_2
B_2
B'_2
A_3
B_3
B'_3
A_{s-1}
B_{s-1}
B'_{s-1}
A_s
$= \ell_1$
$= \ell_2$
$= \ell_3$
$= \ell_{s-1}$
$= \ell_s$

FIGURE 6

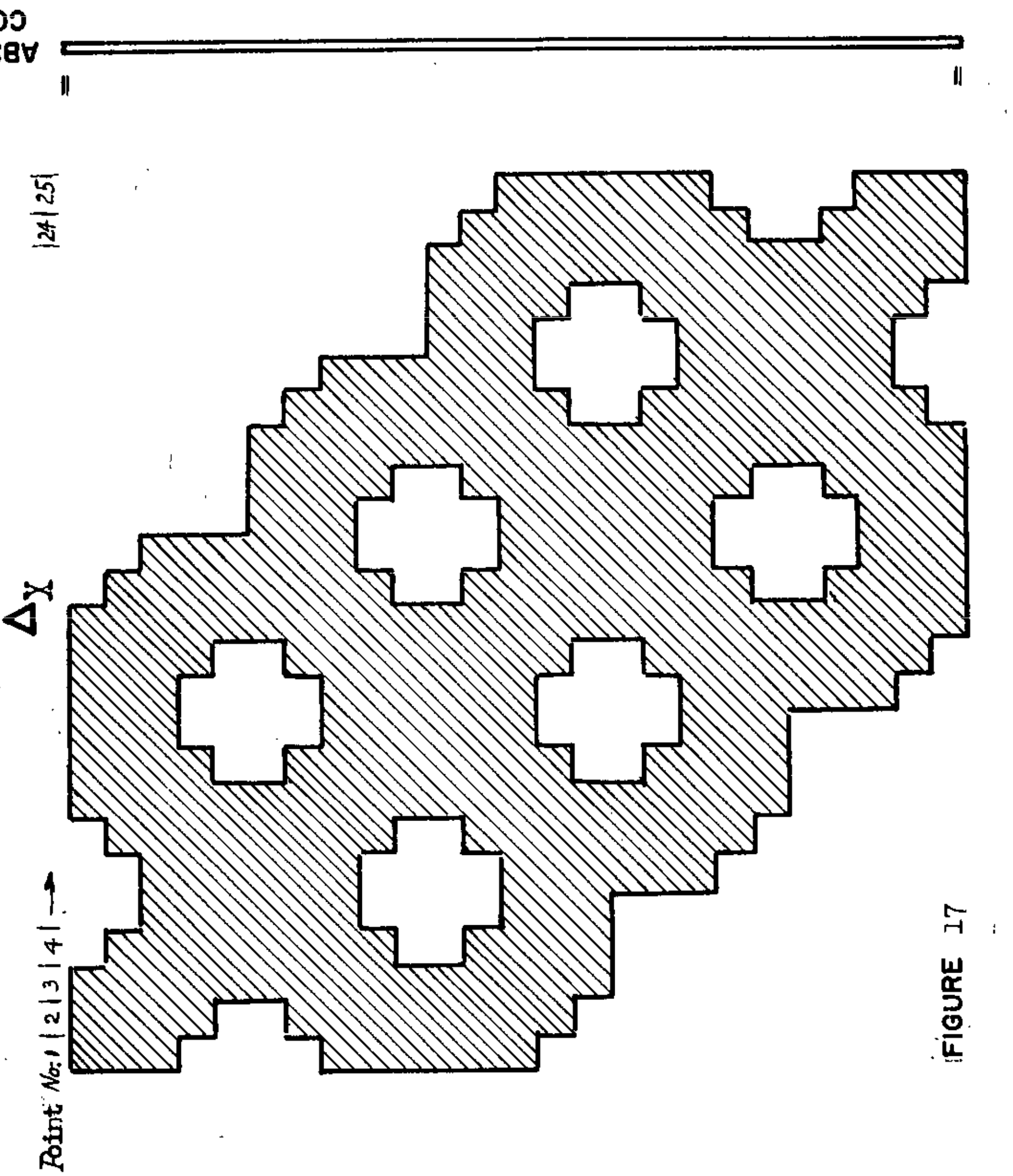


FIGURE 17

Two possibilities for the solution of this problem shall be discussed.

The first solution is rigorous. Its result is influenced only by the propagation of the rounding errors in the computing machine.

The solution depends on the repeated application of formulas (22) - (30), thus eliminating, stepwise, certain groups of unknowns. The system of partitioning as displayed in Fig. 16, is based on the stepwise elimination of groups containing twelve unknowns. This arrangement was chosen in order to minimize the number of zeros in the individual submatrices, denoted by B_i's, thus increasing the economy of the necessary computations. Denoting the s-times reduced system of normal equations in accordance with the well known notation used by Gauss, as .s,

we obtain:

$$\Delta_{o_s} = \left[A_s \cdot (s-1)\right]^{-1} \left[\ell_s \cdot (s-1)\right]$$

$$\Delta_{o_{s-1}} = \left[A_{s-1} \cdot (s-2)\right]^{-1} \left[\ell_{s-1} \cdot (s-2) - B_{s-1} \Delta_{o_s}\right]$$

$$\vdots$$

$$\Delta_{o_1} = A_1^{-1} \left[\ell_1 - B_1 \Delta_{o_2}\right] \tag{127}$$

where

$$A_2 \cdot 1 = A_2 - B_1' A_1^{-1} B_1 \qquad \text{and} \qquad \ell_2 \cdot 1 = \ell_2 - B_1' A_1^{-1} \ell_1$$

$$A_3 \cdot 2 = A_3 - B_2' (A_2 \cdot 1)^{-1} B_2 \qquad \ell_3 \cdot 2 = \ell_3 - B_2' (A_2 \cdot 1)^{-1} \ell_2 \cdot 1$$

$$\vdots$$

$$A_s \cdot (s-1) = A_s - B_{s-1}' \left[A_{s-1} \cdot (s-2)\right]^{-1} B_{s-1} \qquad \ell_s \cdot (s-1) = \ell_s - B_{s-1}' \left[A_{s-1} \cdot (s-2)\right]^{-1} \ell_{s-1} \cdot (s-2)$$

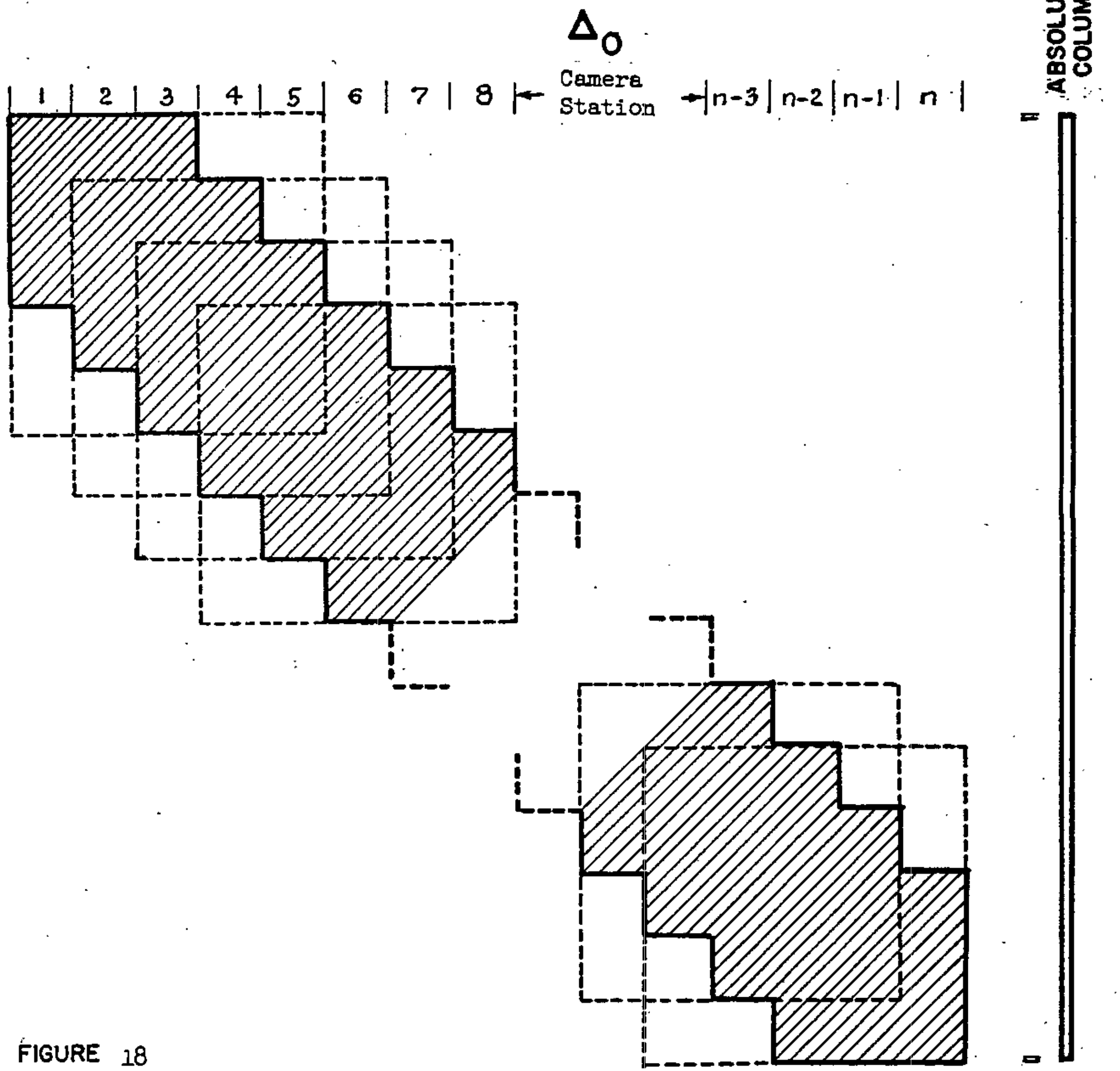


FIGURE 18

The other solution uses a technique related to the Gauss-Seidel relaxation method. A submatrix chosen for geometrical reasons, moves in steps across the original matrix. Once more the escalator matrix of Fig. 16 is seen in Fig. 18 as the shaded area within the heavy contour lines. In a strip flown with 2/3 overlap, five consecutive camera stations are connected by the resulting overlap of the photographs. Therefore, having six unknowns per station, a 30 x 30 submatrix was chosen, which always contains such a group of unknown orientation parameters, as they belong to five consecutive camera stations. This partial system is now displaced along the strip by one station each time. Thus, for a strip with n photographs one obtains (n-4) such submatrices and consequently by inversion of these submatrices one obtains five values for each of the orientation parameters with the exception of the first and last four photographs, where accordingly fewer values are obtained. The arithmetic means of the roots of the individual parameters are now computed and considered as the result of any one iteration cycle. The approximation results thus obtained are used to continue the computation according to the Gauss-Seidel relaxation method, by changing the original absolute column, taking into account the coefficients not incorporated in the individual 30 x 30 submatrices, together with the values of the corresponding orientation parameters, as obtained in the preceding iteration cycle. The iteration is continued until the roots have converged to a pre-established accuracy level. For the economy of the solution, it is of importance that only in the first iteration cycle the individual submatrices must be inverted. The roots in the following cycles are then determined by multiplying the individual inverses by the changed absolute columns.

IX. SUMMARY AND CONCLUSIONS

The analytical solution for the general problem of photogrammetry, as presented in this report, is not restricted by geometrical or statistical considerations, because all nine geometrical parameters which characterize a central perspective can be introduced for any number of photographs. (Compare [16]). Furthermore, provisions have been made to consider all types of measurements, as they may arise, as erroneous. A least squares treatment results in the most probable values of the unknowns of the solution, provided that the residual errors are normally distributed and the various bundles of rays are generated according to the principle of the central perspective.

Physical influences, which deform the central perspective bundles and therefore devaluate the condition of collinearity (compare Chapter III), are lens distortion and refraction. The lens distortion can be determined along with the geometrical parameters (See Chapter VI). Star or collimator photography, in connection with the analytical treatment of an individual photograph, provides a practically unbiased means of determining the calibration of such a camera; that is, its interior orientation and the specific lens distortion. The simultaneously obtained elements of exterior orientation allow the calibration of phototheodolites or the establishment of the mutual relation between several cameras as, e.g., is necessary in connection with the use of a sun camera. (See Chapter IV-C)

Based on practical experience, it can be generally stated that the analytical treatment appears to be an excellent means for analyzing objectively the components of the photogrammetric measuring method.

If refraction can not be eliminated by a suitable arrangement of the measuring set-up, its influence must be eliminated for each individual ray, making use of specific meteorological measurements or assuming a certain model of the atmosphere. Information concerning the computation of the corresponding corrections and their consideration in the analytical reduction method is given in Chapter VII.

Ignoring the economical side of the problem, in precision photogrammetry the requirements for flatness and dimensional stability of the emulsion carrier can be satisfied by the use of precision ground glass plates. The unavoidable irregular shrinkage of the emulsion, together with the measuring errors on the comparator cause the plate coordinate measurements to be affected by residual errors, which have a distribution similar to a normal distribution, thus justifying the effort of a rigorous least squares treatment. Such a computing technique will also prove to be practical in cases where systematic errors are present. Not only does a least squares solution in any case produce the most likely result, but additionally obtained qualitative and quantitative information allows one to recognize and isolate systematic errors. In addition, the least squares treatment, as described in this report, is simple, due to the simplicity of the mathematical model on which the general photogrammetric solution is based. Thus, the least squares solution, can be considered an economical technique for intersecting corresponding rays of various bundles.

A comparison of this report with [2], shows that the present approach distinguishes itself from the former one only by the way in which the elimination of the unknown coordinates of the model X is accomplished. In the present solution, this elimination of unknowns is not performed algebraically before the observation equations are formed but during the process of forming the reduced normal equations. Consequently, the system of reduced normal equations (formula (37)) agrees, to all but second order terms, which have been neglected during the process of linearization by the Taylor series, with the corresponding normal equation system of the formerly published solution. The advantage of the present solution can be seen in the simple and systematic flow of the computations. It is possibel to treat all cases of practical analytical photogrammetry with but one basic computing scheme, thus simplifying considerably the "bookkeeping effort" in the electronic computer.

A critical study of the indivual steps of the presented solution leads to a conclusion which, although somewhat discouraging for the author, may encourage the application of analytical photogrammetry. It becomes obvious that the analytical treatment of photogrammetric problems does not call for any new manipulations in photogrammetric theory or statistical treatment of errors.

The expressions, representing the basis of the whole solution, derived in equations (11) and (12), as they exist between the coordinates of the model and the corresponding coordinates of the images, are the well-known formulas derived by v. Gruber in [14] and the corresponding inverse functions. The partial differential quotients necessary to form the observation equations (15), given with formulas (40) and (41) are the same expressions that are found in [1] or [2]; the unmodified use of these expressions appears justified due to the simplicity of their construction. The system of normal equations (21) resulting from the system of observational equations (18) is identical, as mentioned before, with the solution given by Helment in [4]. The elimination of the vectors, K in the system of normal equations (21), Δ_X in the system (35), and the reduction of the normal equation system for an unlimited strip as shown in Fig. (16), (formula (127)), are performed with a certain sequence of matrix operations, formulas (22) to (30). This technique however, is as shown by Gotthardt in [15] none other than the classic Gaussion elimination for two unknowns, applied to matrix calculus. The alternate solution for solving the system of normal equations of a strip or block triangulation using an iterative approach, is based on the Gauss-Seidel Relaxation method [17], in connection with a procedure which has proven useful in similar applications, known as "Smoothing with Moving Arcs."

Consequently, the knowledge of the classical geometrical considerations dealing with a central projection, and a certain familiarity with the method of least squares adjustment, as applied in geodesy, suffice to solve the analytical problems arising from the application of photogrammetric measuring systems.

As previously mentioned, the system of normal equations (35) schematically shown in Fig. 4, is typical for the most general problem of analytical photogrammetry. The system is readily accumulated because the corresponding observational equations are based on the simplest geometrical model conceivable, expressing for any type of control point the condition of collinearity between control point, center of projection and image point. The problem of arriving at the most economical and feasible method of reduction in analytical photogrammetry is, therefore, concerned with the process of determining the roots of this normal equation system. The solution presented in this report is based on the formation of a system of reduced normal equations (formulas (37) or (38)) by a rigorous mathematical method. The attractiveness of the solution arises from the presence of a series of separated square matrices along the main diagonal. This feature, however, is lost in the resulting system of reduced normal equations, where, quite obviously, all the remaining unknowns are more or less correlated, depending on the geometrical arrangement of the cameras. Any attempt to further simplify the process of determining the roots of the normal equation system (35) as it is readily seen from Fig. 4, must try to preserve all of the fully separated square matrices along the main diagonal during a complete computational cycle. Such a result is obtained if a computational cycle, using the relaxation techniques, is established around the point which separates the unknown parameter corrections associated with the model Δ_X from those associated with the camera orientations Δ_O. Assuming, as a first step, Δ_O a null vector, one obtains a Δ_X vector by a series of inversions of 3x3 matrices, each of which contains the coordinate corrections of a single control point. Multiplying this Δ_X vector with the submatrix, $B_O^T(AP^{-1}A^T)^{-1}B_X$, and adding this result to the absolute column, $B_O^T(AP^{-1}A^T)^{-1}\ell$, a Δ_O vector can be computed by inversions of a series of maximum (9x9) matrices, each of which contains the corrections to the orientation elements of a single camera. This new Δ_O vector is multiplied with the submatrix, $B_X^T(AP^{-1}A^T)^{-1}B_O$, and added to the absolute column, $B_X^T(AP^{-1}A^T)^{-1}\ell$ and a new Δ_X vector is computed using the aforementioned computational steps, which in turn lead to the computation of a new Δ_O vector. This method will converge although very reluctantly. A geometrical analogue of such a solution, although not entirely descriptive, leads to the following approach. Starting with certain approximations for the orientation parameters, sets of coordinates of all points of the model are computed with formulas (56) and (57), which may be subjected to an after-treatment according to formula (39) with Δ_O as null vector. In any case, the maximum size of the matrix which must be inverted is (3x3). With the thus obtained spatial coordinates of all points of the model, a series of resections in space is computed, thus obtaining the orientations of all cameras. In these computations, matrices of (9x9) maximum size must be inverted. With the thus computed orientation parameters, a new set of coordinates of the model are computed, on which a new set of camera orientations can be based. By repeating these two phases, alternately, the final orientations and correspondingly, the final coordinates of all measured points of the model can be computed. Again, the convergence is extremely slow. An increase in the slope of convergence would remedy this situation. The associated numerical effort may be considerable. Such methods are described in [17]. On the other hand the development of electronic computers progresses at an impressive rate, with respect to both storage facilities and computing speeds. In the near future, it should be possible to handle, economically, numerical solutions requiring a very large number of iteration cycles. It is believed that this situation will make possible a solution based on relaxation techniques for the general photogrammetric problem. This remark is made especially with respect to the analytical treatment of an extended block, where the approach as outlined in this report still leads to a rather bulky matrix of normal equations, as shown in Fig. 17.

Concluding, it appears that the problem of analytical photogrammetry today requires concentration not so much on the problem of the numerical treatment of the measured plate coordinates, but more on the technical difficulties associated with problems of identifying and measuring precisely the contents of photographs.

Hellmut H. Schmid

HELLMUT H. SCHMID

REFERENCES

1. Schmid, H. — An Analytical Treatment of the Orientation of a Photogrammetric Camera, Ballistic Research Laboratories Report No. 880, Aberdeen Proving Ground, Maryland, Oct. 1953. An abstract of this report was published in Photogrammetric Engineering, Vol. XX, No. 5, Dec. 1954.

2. Schmid. H. — An Analytical Treatment of the Problem of Triangulation by Stereophotogrammetry, Ballistic Research Laboratories Report No. 961, Aberdeen Proving Ground, Maryland, Oct. 1955. This report was published in Photogrammetria, Vol. XIII, No. 2 & 3, 1956 - 1957.

3. Rinner, K. — Analytische photogrammetrische Triangulation mit formentreuen Bündeln. Reihe A. Höhere Geodäsie Heft Nr. 25, Deutsche Geodätische Kommission. München (1957).

4. Helmert, F. R. — Die Ausgleichsrechnung nach der Methode der kleinsten Quadrate, Leipzig (1872).

5. Schmid, H. — Determination of Spatial Position and Attitude of a Bombing Aircraft by an Airborne Photogrammetric Camera, Ballistic Research Laboratories Memo Report No. 787, Aberdeen Proving Ground, May 1954 and Photogrammetric Engineering, Vol. XXI, No. 1, March 1955.

6. Gotthardt, E. — Gemeinsame Lageeinpassung mehrerer Modelle mittels verketteter Helmerttransformationen, Bildmessung und Luftbildwesen, June 1959

7. Schmid, H. — Some Remarks on the Problem of Transforming Geodetic Ellipsoidal Coordinates into Cartesian Coordinates with the Help of the Reduced Latitude, Ordnance Computer Research Report, Ballistic Research Laboratories, Aberdeen Proving Ground, Maryland, Vol. VI, No. 2, 15 April 1959.

8. Garfinkel, Boris — An Investigation in the Theory of Astronomical Refraction, Astronomical Journal, Vol. 50, No. 8 1944 Feb., pp 169 - 179.

9. Hanson, Fred. S. — Atmospheric Refraction Errors for Optical Instrumentation, Technical Memo, No. 104, Oct. 1953, White Sands Proving Ground, Las Cruces, New Mexico.

REFERENCES

10. Minzner, R. A.; Ripley, W. S.; Condron, T. P. — U. S. Extension to the ICAO Standard Atmosphere, U. S. Dept. of Commerce, Weather Bureau, 1958.

11. Brown, D. C. — A Treatment of Analytical Photogrammetry (With Emphasis on Ballistic Camera Applications) RCA Data Reduction Technical Report No. 39, 20 August 1957.

12. Jordan, Eggert, Kneissl — Handbuch der Vermessungskunde Vol. III, page 410, Stuttgart, 1955.

13. Mallinckrodt — Propagation Errors in Radio Location Systems, Air Force Technical Report No. 4, AFMTC, Patrick Air Force Base, Florida, 1 June 1951.

14. v. Gruber, O. — Ferienkurs in Photogrammetrie, Stuttgart (1930)

15. Gotthardt, E. — Boltzsches Entwicklungsverfahren und Gauszscher Algorithmus, Zeitschrift fur Vermessungswesen, 78, Jahrgang, April 1953, Heft 4.

16. Schmid, H. — Eine allgemeine analytische Lösung für die Aufgabe der Photogrammetrie, Bildmessung und Luftbildwesen, Heft 4/1958 und 1/1959.

17. Stiefel, Eduard — Über einige Methoden der Relaxationsrechnung, Zeitschrift fur angewandte Mathematik und Physik, Vol. III, 15 Jan. 1952

DISTRIBUTION LIST

No. of Copies	Organization
1	Chief of Ordnance Department of the Army Washington 25, D. C. Attn: ORDTB - Bal Sec
2	Commanding Officer Diamond Ordnance Fuze Laboratories Washington 25, D. C. Attn: ORDTL - 012 Mr. Francis E. Washer
10	Director Armed Services Technical Information Agency Arlington Hall Station Arlington 12, Virginia Attn: TIPCR
1	Office of Technical Services Department of Commerce Washington 25, D. C.
10	British Joint Services Mission 1800 K Street, N. W. Washington 6, D. C. Attn: Reports Officer Of Interest to: Prof. E. H. Thompson University College London, W. C. 1 England
5	Canadian Army Staff 2450 Massachusetts Avenue Washington 8, D. C. Of Interest to: National Research Council of Canada Ottawa 2, Ontario Attn: Mr. T. J. Blachut Mr. G. H. Schut
3	Chief, Bureau of Ordnance Department of the Navy Washington 25, D. C. Attn: ReO
2	Commander Naval Ordnance Laboratory White Oak Silver Spring 19, Maryland
1	Commander U. S. Naval Ordnance Test Station China Lake, California
1	Commander Naval Proving Ground Dahlgren, Virginia Attn: Dr. W. A. Kemper
1	Commander U. S. Naval Photographic Interpretation Center Anacostia Station Washington 20, D. C.
1	Commander U. S. Naval Hydrographic Center Washington 25, D. C.
2	Commander U. S. Naval Air Missile Test Center Point Mugu, California
1	Director U. S. Naval Observatory Washington 25, D. C. Attn: Dr. William Markowitz

DISTRIBUTION LIST

No. of Copies	Organization
1	Commander Aeronautical Chart and Information Center Air, Photographic and Charting Service (MATS) Second & Arsenal Sts. St. Louis 18, Missouri Attn: Chief Photogrammetry Research Branch Mr. Edwin Roth
1	Director National Science Foundation Washington, D. C. Attn: Technical Library
3	Director U. S. Geological Survey Washington 25, D. C. Attn: Chief, Topographic Eng. Mr. Russel K. Bean Mr. R. E. Altenhofen
3	Commander ARGMA Redstone Arsenal, Alabama Attn: Dr. Ernst Stuhlinger Dr. Hellmut Holzer Dr. Bruns
1	Commanding General U. S. Army Signal Corps Engineering Labs. Fort Monmouth, New Jersey Attn: Tech Document Center
3	Commanding General White Sands Missile Range New Mexico Attn: ORDBS - TS- TIB Mr. Ted Durrenberger
1	Commander Holloman Air Development Center, SEB, FDL Holloman Air Force Base, New Mexico
5	Commanding General Engineering Research & Development Labs. Fort Belvoir, Virginia Attn: Mr. William C. Cude Mr. D. Esten Mr. Eldon Sewell Mr. John T. Pennington Mr. Matos
4	Commanding Officer Army Map Service 6500 Brooks Lane Washington 16, D. C. Attn: Photogrammetric Division Col. F. O. Diercks Mr. A. Nowicki Mr. J. Theis Mr. Wm. M. Kaula
1	Commanding Officer Office of Ordnance Research Box CM, Duke Station Durham, North Carolina Attn: Dr. Theodore W. Schmidt
1	Director Astro Physical Observatory 60 Garden St. Cambridge 38, Mass. Attn: Dr. Fred Whipple
2	Automatic Corp. Paramount Bldg. N. Y. 36, New York Attn: Dr. Ulrich Heidelauf Mr. E. L. Merritt
1	Bausch & Lomb Optical Co. 262 St. Paul Street Rochester, New York Attn: Mr. Heinz Gruner

DISTRIBUTION LIST

No. of Copies	Organization
1	Director Air University Library Maxwell Air Force Base, Alabama
3	Commander Air Proving Ground Center Eglin Air Force Base, Florida Attn: PGTRI PGEES - Mr. W. A. Brown (2 cys)
2	Commander Air Force Missile Test Center (AFMT Tech Lib. MU-135) Patrick Air Force Base, Florida
2	Commander Air Force Cambridge Research Center, ARDC L. G. Hanscom Field Bedford, Massachusetts Attn: Geophysics Research Directorate
1	Commander Wright Air Development Center Wright-Patterson Air Force Base, Ohio Attn: Aeronautical Research Laboratory
1	Commander Rome Air Development Center Griffiss Air Force Base Rome, New York Attn: Library
1	Commanding Officer Air Force Aeronautical Chart Service Washington, D. C. Attn: Division of Photogrammetry
1	Joint Coordinating Committee on Guided Missiles Office of the Assistant Secretary of Defense (R&D) The Pentagon Washington 25, D. C.
1	National Aeronautics & Space Administration 1520 H. Street, N. W. Washington 25, D. C. Attn: Dr. John O'Keefe
1	NASA Goddard Space Flight Development Center Satellite Application Division 4555 Overlook Avenue, S. W. Washington 25, D. C. Attn: Dr. Nordberg
1	Director National Aeronautics & Space Agency Langley Research Center Langley Field, Virginia Attn: Mr. Richard M. Dickerson
1	Director National Bureau of Standards Washington 25, D. C. Attn: Dr. Irvine C. Gardner
4	Director U. S. Coast & Geodetic Survey Department of Commerce Washington 25, D. C. Attn: Commander L. W. Swanson Mr. G. C. Tewinkel Mr. Charles Whitten Mr. L. G. Simmons

DISTRIBUTION LIST

No. of Copies	Organization
1	Bendix Aviation Corporation Computer Division 5630 Arbor Vitae Street Los Angeles 45, California Attn: V. A. van Praag
1	Cornell University School of Civil Engineering Ithaca, New York Attn: Prof. Arthur J. McNair Center of Aerial Photographic Studies
1	Cincinnati Observatory Cincinnati, Ohio Attn: Dr. Paul Herget
1	International Business Machine Corporation Endicott, New York Attn: Mr. John V. Sharp
1	Mass. Inst. of Tech. Dept. of Civil & Sanitary Eng. Cambridge 39, Mass. Director Photogrammetry Lab. Attn: C. L. Miller
1	New Mexico School of Agriculture & Mechanic Arts State College, New Mexico Attn: Dr. George Gardiner
1	National Academy of Science Washington 25, D. C. Attn: Dr. Pembroke J. Hart, Jr.
3	Ohio State University Mapping and Charting Research Lab. Columbus, Ohio Attn: Prof. Dr. Brandenberger Prof. F. J. Doyle Prof. A. R. Robbins
1	Photogrammetry, Inc. Silver Spring, Maryland Attn: Mr. Gomer T. McNeil
1	Director Physical Research Lab. Boston University Boston, Massachusetts
1	Princeton University Dept. of Civil Eng. Princeton, New Jersey Attn: Prof. Sumner B. Irish
1	RCA Service Co., Incorporated Patrick Air Force Base, Fla. Attn: RCA Data Reduction Gr. Analysis Unit Mr. Duane Brown
1	Ramo - Wooldridge Corporation 5730 Arbor Vitae Street P. O. Box 45067 Airport Station Los Angeles 45, California Attn: Dr. E. Stern
2	Syracuse University L. C. Smith College of Engineering Dept. of Civil Eng. Syracuse 10, New York Attn: Prof. Arthur Faulds Prof. Paul G. Brennan
1	Syracuse University Department of Civil Eng. Syracuse, New York Attn: Photogrammetric Department

DISTRIBUTION LIST

No. of Copies	Organization
2	Smithsonian Institute Washington 25, D. C. Attn: Dr. Karl G. Henize Dr. J. Allen Hynek
1	Technical Operations, Inc. Burlington, Mass. Attn: Mr. Samuel S. Holland
1	Telecomputing Corporation Engineering Services Division Burbank, California
2	University of Illinois Department of Civil Eng. Urbana, Illinois Attn: Dr. H. M. Karara Prof. Milton O. Schmid
1	University of Pennsylvania Computing Lab. Philadelphia, Pa. Attn: Mr. John B. Crozier
1	University of Michigan Ann Arbor, Michigan Attn: Prof. Edward Young
1	University of New York Lamount Geological Laboratory New York, New York Attn: Dr. W. M. Ewing
3	Wild Heerbrugg Instruments, Inc. Main & Covert Streets Port Washington, New York Attn: Mr. K. E. Reynolds
1	Western Development Labs. Philco Corp. 3875 Fabian Way Palo Alto, California Attn: Mr. Eric Besag
5	Zeiss - Aerotopograph 929 Highgate Road Alexandria, Virginia Attn: Mr. Marshall S. Wright
1	Mr. Robert Zurlinden 2118 - 37 th St. N. W. Washington 7, D. C.

AD ____________ Accession No. ____________

UNCLASSIFIED

Ballistic Research Laboratories, APG
A GENERAL ANALYTICAL SOLUTION TO THE PROBLEM OF PHOTOGRAMMETRY Dr. Hellmut H. Schmid

BRL Report No. 1065 July 1959

DA Proj No. 503-06-0110, ORD Proj No. TB3-0538
UNCLASSIFIED Report

Photogrammetry - Analysis
Position finding -
Mathematical Analysis

An analytical treatment of the general problem of photogrammetry is developed which only requires that object point, center of projection and corresponding image point are collinear for any one ray present in a specific photogrammetric measuring procedure. The corresponding formulas, expressing the geometrical relation existing between the spatial coordinates of the object, the plate coordinates of the corresponding image and the elements of orientation, are the bases for a rigorous least squares adjustment, whereby both the image-coordinates measured on a comparator, and the given control data may be considered as erroneous. The corresponding system of normal equations, which is shown to be typical for any photogrammetric measuring problem, is used to form a system of reduced normal equations, the unknowns of which are either the elements of orientation or the spatial coordinates of the model.

AD ____________ Accession No. ____________

UNCLASSIFIED

Ballistic Research Laboratories, APG
A GENERAL ANALYTICAL SOLUTION TO THE PROBLEM OF PHOTOGRAMMETRY Dr. Hellmut H. Schmid

BRL Report No. 1065 July 1959

DA Proj No. 503-06-0110, ORD Proj No. TB3-0538
UNCLASSIFIED Report

Photogrammetry - Analysis
Position finding -
Mathematical Analysis

An analytical treatment of the general problem of photogrammetry is developed which only requires that object point, center of projection and corresponding image point are collinear for any one ray present in a specific photogrammetric measuring procedure. The corresponding formulas, expressing the geometrical relation existing between the spatial coordinates of the object, the plate coordinates of the corresponding image and the elements of orientation, are the bases for a rigorous least squares adjustment, whereby both the image-coordinates measured on a comparator, and the given control data may be considered as erroneous. The corresponding system of normal equations, which is shown to be typical for any photogrammetric measuring problem, is used to form a system of reduced normal equations, the unknowns of which are either the elements of orientation or the spatial coordinates of the model.

AD ____________ Accession No. ____________

UNCLASSIFIED

Ballistic Research Laboratories, APG
A GENERAL ANALYTICAL SOLUTION TO THE PROBLEM OF PHOTOGRAMMETRY Dr. Hellmut H. Schmid

BRL Report No. 1065 July 1959

DA Proj No. 503-06-0110, ORD Proj No. TB3-0538
UNCLASSIFIED Report

Photogrammetry - Analysis
Position finding -
Mathematical Analysis

An analytical treatment of the general problem of photogrammetry is developed which only requires that object point, center of projection and corresponding image point are collinear for any one ray present in a specific photogrammetric measuring procedure. The corresponding formulas, expressing the geometrical relation existing between the spatial coordinates of the object, the plate coordinates of the corresponding image and the elements of orientation, are the bases for a rigorous least squares adjustment, whereby both the image-coordinates measured on a comparator, and the given control data may be considered as erroneous. The corresponding system of normal equations, which is shown to be typical for any photogrammetric measuring problem, is used to form a system of reduced normal equations, the unknowns of which are either the elements of orientation or the spatial coordinates of the model.

AD ____________ Accession No. ____________

UNCLASSIFIED

Ballistic Research Laboratories, APG
A GENERAL ANALYTICAL SOLUTION TO THE PROBLEM OF PHOTOGRAMMETRY Dr. Hellmut H. Schmid

BRL Report No. 1065 July 1959

DA Proj No. 503-06-0110, ORD Proj No. TB3-0538
UNCLASSIFIED Report

Photogrammetry - Analysis
Position finding -
Mathematical Analysis

An analytical treatment of the general problem of photogrammetry is developed which only requires that object point, center of projection and corresponding image point are collinear for any one ray present in a specific photogrammetric measuring procedure. The corresponding formulas, expressing the geometrical relation existing between the spatial coordinates of the object, the plate coordinates of the corresponding image and the elements of orientation, are the bases for a rigorous least squares adjustment, whereby both the image-coordinates measured on a comparator, and the given control data may be considered as erroneous. The corresponding system of normal equations, which is shown to be typical for any photogrammetric measuring problem, is used to form a system of reduced normal equations, the unknowns of which are either the elements of orientation or the spatial coordinates of the model.